

On relative K -motives, weights for them, and negative K -groups

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Abstract

In this paper we study certain triangulated categories of K -motives $\mathcal{DK}(-)$ over a wide class of “nice enough” base schemes, and define certain “weights” for them (so, we introduce Chow weight structures for K -motives closely related to the ones introduced by D. Hebert, the first author, and M. Ivanov for “relative Voevodsky motives” of various types). We relate the weights of particular K -motives (of the form $f_*(\mathbf{1}_Y)$, where $\mathbf{1}_Y$ is the tensor unit of $\mathcal{DK}(Y)$) to (negative) homotopy invariant K -groups (tensored by $\mathbb{Z}[\mathcal{S}^{-1}]$ for \mathcal{S} being the set of “non-invertible primes”) $\mathcal{K}_*(-)$. Our results yield a (new) result on the vanishing of $\mathcal{K}_i(Y)$ and of certain relative \mathcal{K} -groups for i being “too negative”; this statement is closely related to a question of Ch. Weibel. We also prove that $\mathcal{K}_i(Y)$ for $i < 0$ is “supported in codimension $-i$ ”. Moreover, we establish several criteria for bounding (below) the weights of the K -motives $f_*(\mathbf{1}_Y)$; this automatically implies the vanishing of the corresponding E_2 -terms of *Chow-weight spectral sequences* (and of the factors of the corresponding Chow-weight filtrations) for any (co)homology of these motives. Our methods of bounding weights use a resolution of singularities result of O. Gabber. They can be applied to various “motivic” triangulated categories; this yields some new statements on (constructible) complexes of étale sheaves (as well as similar bounds on the terms of Chow-weight spectral sequences for Voevodsky motives).

We also relate the weights of K -motives with rational coefficients to that of Beilinson motives; the Chow-weight spectral sequences converging to their \mathbb{Q}_l -étale (co)homology yield Deligne-type weights for

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the latter. Somewhat surprisingly, we are able to prove in certain (“extreme”) cases that the corresponding weight bounds coming from étale (co)homology are precise; we illustrate these statements by some simple examples.

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Introduction

In [Wei80] Ch. Weibel has asked whether $K_{-n}(X) = \{0\}$ for any noetherian scheme X of dimension less than n (where K_* denotes the Bass–Thomason–Trobaugh K -theory). Recall here that negative K -groups of regular schemes vanish; so the question is related to the singularities of X . The positive answer to Weibel’s question for schemes essentially of finite type over a field of characteristic zero was given in [CHSW08]. The next step in this direction was made in [Kel12] where (essentially) the so-called homotopy invariant K -theory was considered;¹ this result was improved in [KeS17].

In the current paper we propose a “categorification” of this question, and relate negative (homotopy invariant) K -groups to weights of K -motives (similar to the weights constructed in [Heb11], [Bon14], and [BoI15] for the corresponding versions $DM(-)$ of the Voevodsky motivic categories following the pattern introduced in [Bon10a]). For this purpose we consider the homotopy categories of modules over the symmetric motivic ring spectra KGL'_- (instead of modules over the motivic cohomology spectra); we localize them by inverting all the primes that are “not invertible over our base schemes” (for this set of primes \mathcal{S} the localization $\mathbb{Z}[\mathcal{S}^{-1}]$ will be denoted by Λ) to obtain our “main” motivic categories $\mathcal{DK}(-)$. The version of K -theory “corresponding to” categories of this type is the homotopy invariant K -theory tensored by Λ that is denoted by $\mathcal{K}(-)$ below.

Voevodsky himself started developing a four functor formalism for his motivic categories (first constructed in [Voe96] and later developed in [VSF00] and [MV99]). Part of this effort was described by Deligne in [Del01], and later the program was carried on by Ayoub in [Ayo07]. In our case, the Voevodsky’s four functor formalism (as developed in the treatise [CiD12], which relied on [Ayo07]) yields exact functors f_* and $f_! : \mathcal{DK}(Y) \rightarrow \mathcal{DK}(X)$ for any separated finite type morphism $f : Y \rightarrow X$ (as well as f^* and $f^! : \mathcal{DK}(X) \rightarrow \mathcal{DK}(Y)$). The properties of these functors and categories (along with the vanishing of negative \mathcal{K} -theory of fields and regular schemes) allow us to define the *Chow weight structure* for $\mathcal{DK}(S)$ for S being a (Λ) -*nice* scheme (see Remark 1.1.4(1) below). Somewhat similarly to [BoI15], the non-negative K -motives $\mathcal{DK}(S)_{w_{\mathrm{chow}} \geq 0}$ over S are “generated by” $v_*(\mathbf{1}_V)$

¹The reason for this was the usage of *cdh*-descent provided by [Cis13] for homotopy invariant K -theory; note that it fails for the “usual” K -theory (of singular schemes). Also, the usage of Gabber’s resolution of singularities result has forced Kelly and us to invert all the positive residue field characteristics of X in the coefficient ring. Yet note that our results certainly yield certain properties of K -theory when restricted to schemes of fixed positive characteristic; see Remark 3.3.2(4) below and Theorem 3.7 of [Kel14] for more detail.

for $v: V \rightarrow S$ running through finite type separated morphisms with regular domain, whereas $\mathcal{DK}(S)_{w_{Chow} \leq 0}$ is “generated by” $v_!(\mathbf{1}_V)$, where $\mathbf{1}_V$ denotes the tensor unit object of $\mathcal{DK}(V)$.²

Now, some of the properties of the Chow weight structure on $\mathcal{DK}(-)$ are quite similar to its DM -versions (as described in the aforementioned papers); furthermore, some of them are immediate consequences of the general formalism of weight structures. Certainly, for any Λ -nice S the weight structure $w_{Chow}(S)$ yields weight filtrations and weight spectral sequences $T_{w_{Chow}}(H, -)$ for any (co)homological functor H that factors through $\mathcal{DK}(S)$ (that converge to $H_*(M)$ for $M \in \text{Obj } \mathcal{DK}(S)$). In the case where S is (separated) of finite type over field (whose characteristic is invertible in Λ ; for $\Lambda = \mathbb{Q}$ it suffices to assume that S is of finite type over an excellent noetherian scheme of dimension at most 3) these spectral sequences relate the cohomology of M to that of (the $\mathcal{DK}(S)$ -versions of) Chow motives over S . The latter are the $\mathcal{DK}(X)$ -retract of objects of the form $v_!(\mathbf{1}_V)$ for $v: V \rightarrow S$ being a projective morphism with regular domain (see Remark 2.1.3(3) below). Moreover (for arbitrary S) the “weights” of M are “detected” by $E_2^{**}(T_{w_{Chow}}(H, M))$ (that are $\mathcal{DK}(S)$ -functorial in M) ; see Corollary 1.4.3 below. Furthermore, in the case $\Lambda = \mathbb{Q}$ the weights of K -motives can be expressed in terms of the weights of the “associated” Beilinson motives (i.e., of the objects of the \mathbb{Q} -linear version of $DM(S)$ obtained from M via the functor $\mathcal{M}_{per}(S)$; see Proposition 4.1.1 below). Next, certain conjectures on the so-called mixed motivic sheaves predict that it suffices to consider H being the corresponding *perverse étale homology* functor to compute the weights of the Beilinson motives obtained (see [Bon15] where this functor was treated in detail and the aforementioned conjectures were related to “standard” motivic conjectures over universal domains). We succeed in establishing (a form of) this “weight-detection” conjecture in some particular cases that are rather important for the purposes of the current paper in §4.2–4.3; we illustrate the relevance of our weight bounds by certain simple examples.

Let us now describe the main motivation for treating K -motives in this paper. The orthogonality axiom of weight structures yields for any $n \geq 0$ that an object M belongs to $\mathcal{DK}(X)_{w_{Chow} \geq -n}$ if and only if $\mathcal{DK}(X)_{w_{Chow} \leq -n-1} \perp M$ (i.e., there are only zero morphisms between objects of weights less than n into M). Now, for $M = f_*(\mathbf{1}_Y)$ (where $f: Y \rightarrow X$ is a separated morphism of Λ -nice schemes) these conditions are equivalent to the following one: for

²One may consider only regular V/S or all finite type separated S -schemes when “generating” $\mathcal{DK}(S)_{w_{Chow} \leq 0}$. The main distinctions of our weights for $\mathcal{DK}(-)$ from the weight structure construction used in [BoI15] is that we don’t have to “twist” the generators due to the fact that K -motives are $-(1) = -(1)[2]$ -periodic (see Remark 1.2.2(8) below); moreover, we treat non-compact motives in much more detail than in *ibid*.

any separated finite type morphism $P \rightarrow X$ the groups $\mathcal{K}_i^{BM,Y}(P \times_X Y)$ (see Remark 1.2.2(1)) vanish for $i < -n$; moreover, it suffices to consider those P that are affine over X and regular in this criterion. Here $\mathcal{K}_i^{BM,Y}(-)$ are certain Borel–Moore \mathcal{K} -groups; hence the latter condition is equivalent to the vanishing of $\mathcal{K}_i(U \times_X Y)$ for all smooth U/X and $i < -n$ combined with the surjectivity of the natural homomorphism $\mathcal{K}_{-n}(U \times_X Y) \rightarrow \mathcal{K}_{-n}(V \times_X Y)$ for any such U and any open (dense) $V \subset U$. Thus we obtain a close relation between negative \mathcal{K} -groups and the “weights” of objects of the type $f_*(\mathbf{1}_Y)$; the aforementioned motivic conjectures (along with the particular cases in which we can prove them) yield quite unexpected relations of these matters with the weight spectral sequences converging to étale (co)homology (see §4.3). Note here that there certainly exist connecting “regulator” homomorphisms between \mathcal{K} -groups with rational coefficients and \mathbb{Q}_l -étale cohomology; yet these maps are very far from being surjective or injective (in general).

Next, to obtain results related to the Weibel’s question we bound below the weights of $\mathbf{1}_X$ and of $f_*(\mathbf{1}_Y)$; we also study the Chow-weight decompositions of these objects. It is no wonder that this requires certain resolution of singularities results (note that $f_*(\mathbf{1}_Y) \in \mathcal{DK}(X)_{w_{Chow} \geq 0}$ whenever Y is regular). To make our weight bound results as general as possible, we consider a form of Gabber’s resolution of singularities results that was previously used in [Kel14]. We use an argument closely related to the ones that were used in [ILO14, §XIII.3], [CiD16, §6.2], and [CiD12, §4.2] for the proof of the constructibility of the corresponding “analogues” of $M = f_*(\mathbf{1}_Y)$ for f being of finite type. Now, the reasonings in the aforementioned papers are far from being short and simple, whereas we are interested in much more precise “information” on M . So, we use somewhat technical definitions (including a modification of Gabber’s dimensions functions) and arguments to put objects of the type $f_*(\mathbf{1}_Y)$ and $f_!(\mathbf{1}_Y)$ into certain *envelopes* (i.e., we describe more or less explicit sets of objects of $\mathcal{DK}(X)$ such that the motives that interest us can be obtained from them by means of extensions and retracts). It does not make sense to formulate these results in the introduction; so we will try to describe some of their consequences instead (cf. also Remark 3.4.3(4)). We prove that for any $s \geq 0$ there exists a closed scheme $Z \subset X$ such that $\dim(X) - \dim(Z) \geq s + 1$ and for $U = X \setminus Z$ we have $\mathbf{1}_U \in \mathcal{DK}(U)_{w \geq -s}$. According to the aforementioned relation of $\mathcal{DK}(X)$ -weights of $\mathbf{1}_X = \text{id}_{X,*}(\mathbf{1}_X)$ to \mathcal{K} -groups, it follows that \mathcal{K}_{-s} is “supported in codimension s ”; this statement appears to be completely new (see Remark 3.3.2(2)). We also establish some more criteria for $f_*(\mathbf{1}_Y)$ to belong to $\mathcal{DK}(X)_{w_{Chow} \geq -n}$ (for f being of finite type). In particular (see Theorem 3.3.1(II.2)), it suffices to verify that the groups $\mathcal{K}_{-i}^{BM,Y}(P \times_X Y)$ vanish for $i < -n$ only in the case where P is

regular and affine and $\dim(P) \leq \bar{d} - i$, where \bar{d} is the dimension of an X -compactification of Y .³ Furthermore, it suffices to compute these \mathcal{K} -groups for a finite number of “test schemes” that can be (more or less) explicitly described in terms of f ; see Remark 3.3.2(3) for more detail. Note also that the reader willing to avoid the complicated arguments of §3.2 may replace most of them (including the construction of the aforementioned “test schemes”) by a much easier Proposition 4.5.1 in the case where X is of finite type over a field. On the other hand, in §3.4 the K -motivic “envelope” formulations of §3.2 were extended to a wide range of “motivic” categories; in particular (applying them to étale motives as studied in [CiD16]) one obtains certain “envelope” statement that appear to be quite new over “general” schemes (yet we do not treat this matter in detail).

Let us now describe the contents of the paper. Some more information of this sort can be found at the beginnings of sections.

In §1 we describe some basics on triangulated categories $\mathcal{DK}(-)$ of K -motives over (excellent separated Noetherian finite dimensional) schemes. Since the properties of these categories are quite similar to the ones of “more or less usual” relative Voevodsky motives (as studied in several papers of Ayoub, Cisinski, and Déglise, we only sketch their proofs. We also recall those aspects of the theory of weight structures that will be needed below.

In §2 we introduce our Chow weight structures on $\mathcal{DK}(-)$ (starting from its restriction to the subcategory of compact motives) and study their properties. We also formulate several criteria for the motif $f_*(\mathbf{1}_Y)$ for $f: Y \rightarrow X$ to be of weights $\geq -n$ in $\mathcal{DK}(X)$; they are formulated in terms of negative homotopy invariant K -groups of certain Y -schemes. A remarkable distinction of this section from the study of the Chow weight structures in [Heb11], [Bon14], and [BoI15] is that we study (in detail) the weights of non-compact motives (also); this enables us not to restrict our criteria to the case where f is of finite type.⁴

In §3 we study these criteria along the weight bounds for $f_*(\mathbf{1}_Y)$ in more detail. For this purpose we apply the resolution of singularities results of Gabber for putting the objects $f_*(\mathbf{1}_Y)$ and $f_!(\mathbf{1}_Y)$ (for $f: Y \rightarrow X$ being of finite type) into the envelope of $v_{b,*}\mathbf{1}_{V_b}[m_b]$ for certain finite type morphisms $v_b: V_b \rightarrow X$ with regular domains and $m_b \geq 0$; the pairs (V_b, m_b) may be (more or less) explicitly described. These results can be extended to a wide

³Recall that X -compactifications of any finite type separated Y/X exist, and their dimensions are equal. Moreover, \bar{d} equals $\dim(Y)$ whenever X is of finite type over a field or over $\text{Spec } \mathbb{Z}$; see Remark 3.1.3 below for more detail.

⁴Note also that certain Chow weight structures on “big” motivic categories were the main subject of [BoK17]; however, the reason to consider them was to avoid the usage of resolution of singularities statements.

range of “relative motivic” categories; so our methods yield (in particular) a more precise version of the constructibility of “direct images” of (tensor) unit h -motives as proved in [CiD16, §6.2] following the pattern of [ILO14, §XIII.3] (and it implies a new property of the corresponding constructible complexes of étale sheaves). We bound below the weights of $f_*(\mathbf{1}_Y)$; this yields vanishing results closely related to the (“homotopy invariant” version of the) question of Ch. Weibel on the vanishing of the K -theory of Y in degrees less than $-\dim(Y)$. One may say that our weight bounds are certain “categorifications” of this vanishing question.

In §4 we prove that the “weights” of K -motives can be expressed in terms of the ones of the associated Beilinson motives. Then we proceed to study the weights of the latter (as well as of some “similar” motivic categories); we mostly study the case where the base scheme X is the spectrum of a field or a variety. Under this restriction the weight zero X -motives come from regular projective X -schemes; so the corresponding weight spectral sequences $T_{w_{Chow}(X)}(H, M)$ relate the (co)homology H_* of X -motives to that of X -schemes of this sort. Note here that (according to the general theory of weight structures) the non-vanishing of $E_2^{pq}T_{w_{Chow}(X)}(H, M) \neq 0$ for some $q \in \mathbb{Z}$ and $p > n$ implies that M “contains weights less than $-n$ ”. When $X = \text{Spec } k$ this yields a close connection of the weights of X -motives and schemes with Deligne’s weights of their étale (co)homology; for X being a variety one can use the weights from [BBD82] and [Hub97] here. We have an inequality between these two versions of weight bounds; it is conjecturally an equality, and we prove this conjecture for certain “(almost) maximally singular” X -schemes. We also illustrate our notion of *maximally singular schemes* (this is a certain “motivic” characterization of the singularities of X) by some easy examples. Lastly; we describe some simple substitute of the “motivic resolution of singularities” arguments of §3 in the case where X is a variety; this suggests a close relation of our weight filtrations and spectral sequences to the *singularity* ones considered in [PaP09] and [CiG14] in the characteristic 0 case.

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1 Some preliminaries: notation, K -motives, and weight structures

The results of these section are (more or less) easy consequences of the formalism of (relative) motivic categories and weight structures (as developed

in [CiD12] and [Bon10a], respectively).

In §1.1 we introduce some notation and conventions for (mostly, triangulated) categories. We also recall some basics on “localizing coefficients” in triangulated categories and recall the main result of [BoS15a] (that describes the envelope of a set of object of \underline{C} in terms of cohomological functors from \underline{C}).

In §1.2 we introduce our version of the categories $\mathcal{DK}(-)$ of (relative) K -motives. Our results are essentially the KGl-module versions of the corresponding results of [CiD12] and [CiD16]; so we just skip most of the proofs.

In §1.3 we recall some basics properties on weight structures and prove a few new statements. In §1.4 we introduce the (more complicated) notions of weight complexes and weight spectral sequences; a reader only interested in §2–3 may skip this section.

1.1 Some notation, definitions, and auxiliary statements

For a category \underline{C} the symbol \underline{C}^{op} will denote its opposite category.

For a category \underline{C} , $X, Y \in \text{Obj } \underline{C}$, $\underline{C}(X, Y)$ is the set of \underline{C} -morphisms from X to Y . We will say that X is a *retract* of Y if id_X can be factored through Y . Note that if \underline{C} is triangulated or abelian then X is a retract of Y if and only if X is its direct summand.

For categories $\underline{C}, \underline{D}$ we write $\underline{D} \subset \underline{C}$ if \underline{D} is a full subcategory of \underline{C} .

For any $\underline{D} \subset \underline{C}$ the subcategory \underline{D} is called *Karoubi-closed* in \underline{C} if it contains all retracts of its objects in \underline{C} . We will call the smallest Karoubi-closed subcategory of \underline{C} containing \underline{D} the *Karoubi-closure* of \underline{D} in \underline{C} ; sometimes we will use the same term for the class of objects of the Karoubi-closure of a full subcategory of \underline{C} (corresponding to some subclass of $\text{Obj } \underline{C}$).

The *Karoubi envelope* $\text{Kar}(\underline{D})$ (no lower index) of an additive category \underline{D} is the category of “formal images” of idempotents in \underline{D} .

In this paper all complexes will be cohomological, i.e., the degree of all differentials is $+1$; respectively, we will use cohomological notation for their terms. $K(B)$ will denote the homotopy category of complexes over an additive category B ; for $n \in \mathbb{Z}$ the notation $K(B)^{\leq n}$ is used to denote the class of complexes isomorphic (i.e., homotopy equivalent to) complexes concentrated in degrees $\leq n$.

For a \underline{C} -morphism $f: X \rightarrow Y$ the symbol $\text{Cone}(f)$ denotes the third vertex of the triangle $X \rightarrow Y \xrightarrow{f} \text{Cone}(f) \rightarrow X[1]$ (so, $\text{Cone}(f)$ is well-defined up to a non-canonical isomorphism).

The symbols \underline{C} and \underline{D} below will always denote some triangulated categories. We will use the term “exact functor” for a functor of triangulated

categories (i.e., for a functor that preserves the structures of triangulated categories).

For any distinguished triangle $A \rightarrow B \rightarrow C$ in $\underline{\mathcal{C}}$ we will call B a ($\underline{\mathcal{C}}$ -) *extension* of C by A . A class $D \subset \text{Obj } \underline{\mathcal{C}}$ will be called *extension-closed* if D contains 0 as well as all extensions of its elements by its elements. In particular, an extension-closed D is strict (i.e., contains all objects of $\underline{\mathcal{C}}$ isomorphic to its elements).

The smallest extension-closed D containing a given $D' \subset \text{Obj } \underline{\mathcal{C}}$ will be called the *extension-closure* of D' .

The smallest extension-closed Karoubi-closed subclass of $\text{Obj } \underline{\mathcal{C}}$ containing D (resp. containing $\cup_{i>0} D[i]$, resp. containing $\cup_{i<0} D[i]$) will be called the *envelope* (resp. the *left envelope*, resp. the *right envelope*) of D .

Below $\underline{\mathcal{A}}$ will always denote some abelian category.

We will call a covariant (resp. contravariant) additive functor $H: \underline{\mathcal{C}} \rightarrow \underline{\mathcal{A}}$ *homological* (resp. *cohomological*) if it converts distinguished triangles into long exact sequences.

For $X, Y \in \text{Obj } \underline{\mathcal{C}}$ we will write $X \perp Y$ if $\underline{\mathcal{C}}(X, Y) = \{0\}$. For $D, E \subset \text{Obj } \underline{\mathcal{C}}$ we write $D \perp E$ if $X \perp Y$ for all $X \in D, Y \in E$. For $D \subset \underline{\mathcal{C}}$ the symbol D^\perp denotes the class

$$\{Y \in \text{Obj } \underline{\mathcal{C}} \mid X \perp Y \ \forall X \in D\}.$$

Sometimes we will use the notation D^\perp to denote the corresponding full subcategory of $\underline{\mathcal{C}}$. Dually, ${}^\perp D$ is the class $\{Y \in \text{Obj } \underline{\mathcal{C}} \mid Y \perp X \ \forall X \in D\}$.

We will say that some $C_i \in \text{Obj } \underline{\mathcal{C}}, i \in I$, *Hom-generate* $\underline{\mathcal{C}}$ if for $X \in \text{Obj } \underline{\mathcal{C}}$ we have: $\underline{\mathcal{C}}(C_i[j], X) = \{0\} \ \forall i \in I, j \in \mathbb{Z} \implies X = 0$ (i.e., if $\{C_i[j] \mid j \in \mathbb{Z}\}^\perp$ contains only zero objects).

$M \in \text{Obj } \underline{\mathcal{C}}$ will be called *compact* if the functor $\underline{\mathcal{C}}(M, -)$ commutes with all small coproducts (we will only consider compact objects in those categories that are closed with respect to arbitrary small coproducts).

We will say that a triangulated category $\underline{\mathcal{C}}$ (closed with respect to arbitrary small coproducts) is *compactly generated* if the (triangulated) subcategory of compact objects in it is essentially small and its objects Hom-generate $\underline{\mathcal{C}}$.

For a set of objects $C_i \in \text{Obj } \underline{\mathcal{C}}, i \in I$, we will use the notation $\langle C_i \rangle$ to denote the smallest strictly full triangulated subcategory containing all C_i ; for $D \subset \underline{\mathcal{C}}$ we will write $\langle D \rangle$ instead of $\langle \text{Obj } D \rangle$. We will call the Karoubi-closure of $\langle C_i \rangle$ in $\underline{\mathcal{C}}$ the triangulated category generated by C_i (recall that it is triangulated indeed).

If $\underline{\mathcal{C}}$ is closed with respect to all small coproducts and $\underline{\mathcal{D}} \subset \underline{\mathcal{C}}$ ($\underline{\mathcal{D}}$ is a triangulated category that may be equal to $\underline{\mathcal{C}}$) we will say that the objects C_i

of \underline{D} generate \underline{D} as a localizing subcategory of \underline{C} if \underline{D} is the smallest full strict triangulated subcategory of \underline{C} that contains C_i and is closed with respect to all small coproducts (it easily follows that C_i also Hom-generate \underline{D}).

We list the main properties of the “localization of coefficients” functors for compactly generated triangulated categories.

Proposition 1.1.1. Let \underline{C} be a triangulated category that is compactly generated by its (full) triangulated subcategory \underline{C}' ; let $\mathcal{S} \subset \mathbb{Z}$ be a set of prime numbers. Denote $\mathbb{Z}[\mathcal{S}^{-1}]$ by Λ ; denote by $\underline{C}_{\mathcal{S}\text{-tors}}$ the localizing subcategory of \underline{C} generated by cones of $c \xrightarrow{\times s} c$ for $c \in \text{Obj } \underline{C}$, $s \in \mathcal{S}$.

Then the following statements are valid.

1. $\underline{C}_{\mathcal{S}\text{-tors}}$ is (also) Hom-generated by $c' \xrightarrow{\times s} c'$ for $c' \in \text{Obj } \underline{C}'$, $s \in \mathcal{S}$.
2. The Verdier quotient category $\underline{C}[\mathcal{S}^{-1}] = \underline{C}/\underline{C}_{\mathcal{S}\text{-tors}}$ exists (i.e., the morphism groups of the target are sets) and is closed with respect to small coproducts. Moreover, the localization functor $l: \underline{C} \rightarrow \underline{C}[\mathcal{S}^{-1}]$ respects all coproducts, converts compact objects into compact ones, and $\underline{C}[\mathcal{S}^{-1}]$ is generated by $l(\text{Obj } \underline{C}')$ as its own localizing subcategory.
3. For any $c \in \text{Obj } \underline{C}$, $c' \in \text{Obj } \underline{C}'$, we have $\underline{C}[\mathcal{S}^{-1}](l(c'), l(c)) \cong \underline{C}(c, c') \otimes_{\mathbb{Z}} \Lambda$.
4. $\underline{C}[\mathcal{S}^{-1}]$ is an Λ -linear category.

Proof. See Proposition A.2.8 and Corollary A.2.13 of [Kel12] (cf. also Proposition 5.6.2(I) of [Bon16b] and Appendix B of [Lev13]).

□

Remark 1.1.2. 1. Sometimes we will have to “increase \mathcal{S} ”. For an $\mathcal{S}' \supset \mathcal{S}$ we certainly have obvious exact (localization) comparison functor $\underline{C}[\mathcal{S}^{-1}] \rightarrow \underline{C}[\mathcal{S}'^{-1}]$ that respects compact objects and coproducts. Note that one can obtain it by setting the “new starting category” being equal to $\underline{D} = \underline{C}[\mathcal{S}^{-1}]$.

2. In the case $\mathcal{S}' = \mathbb{P} \setminus \{l\}$ (for $l \in \mathbb{P} \setminus \mathcal{S}$) the corresponding comparison functor will be denoted by $c_{\underline{D}}^l$. Certainly, Proposition 1.1.1(3) yields that the restriction of $c_{\underline{D}}^l$ to the subcategory \underline{D}' of compact objects is the “naive $\mathbb{Z}_{(l)}$ -linearization” functor, i.e., it is the exact functor that tensors the morphism groups by $\mathbb{Z}_{(l)}$.

The following obvious modification of the main result of [BoS15a] appears to be quite useful for the “control of envelopes”.

Proposition 1.1.3. Let $\underline{\mathcal{C}}$ be a small Λ -linear triangulated category, $D \cap \{M\} \subset \text{Obj } \underline{\mathcal{C}}$. Then M belongs to the envelope of D if and only if for any homological functor $F: \underline{\mathcal{C}} \rightarrow \text{Ab}$ we have $F(M) = \{0\}$ whenever the restriction of F to D is zero. Moreover, if $\Lambda \neq \mathbb{Q}$ then it suffices to verify this condition under the assumption that the target of F is $\mathbb{Z}_{(l)}$ -modules for some $l \in \mathbb{P} \setminus \mathcal{S}$.

Proof. The first part of the assertion is immediate from Theorem 0.1 of [BoS15a] (applied to the category $\underline{\mathcal{C}}^{op}$). To obtain the “moreover” part one should note (similarly to *ibid.*) that all the functors $F(-) \otimes_{\mathbb{Z}} \mathbb{Z}_{(l)}$ for $l \in \mathcal{S}$ are cohomological, and we have $F(M) = \{0\}$ whenever $F(M) \otimes_{\mathbb{Z}} \mathbb{Z}_{(l)} = \{0\}$ for all $l \in \mathcal{S}$. \square

Throughout the paper we will only consider schemes that are excellent separated of finite Krull dimension; we will call schemes that satisfy all of these conditions nice ones (for the sake of brevity). All the morphisms we will consider will be separated; they will also usually be of finite type.

Remark 1.1.4. 1. . Moreover, we will usually fix a set of primes \mathcal{S} (and set $\Lambda = \mathbb{Z}[\mathcal{S}^{-1}]$ as in Proposition 1.1.1 above). We will say that a (nice) scheme X is Λ -nice if all the primes in $\mathbb{P} \setminus \mathcal{S}$ are invertible on it (so, the characteristics of all the residue fields of X belong to $\mathcal{S} \cup \{0\}$; in particular, all nice schemes are \mathbb{Q} -nice with $\mathcal{S} = \mathbb{P}$). By default, all the schemes we consider will be Λ -nice also (so, we will assume that a scheme is Λ -nice if it will not be said explicitly that is just nice).

2. . Our reason for concentrating on Λ -nice schemes is that this restriction is necessary for the application of Gabber’s resolution of singularities result (see Remark 3.2.6 below) for the “control of the compactness” of Λ -linear motives (see Theorem 1.2.1(II.4)).

All morphisms of schemes we consider will be separated. We will say that a morphism is smooth only if it is also of finite type.

The following versions of K -theory will be one of the main subjects of this paper: for a scheme X we will use the notation $KH_*(X)$ for the (Weibel’s) homotopy invariant K -theory of X (cf. [Cis13], [Kel14]); $\mathcal{K}(X) = KH(X) \otimes_{\mathbb{Z}} \Lambda$.

We will say that a projective system $S_i, i \in I$, of schemes is *essentially affine* if the transition morphisms $g_{ji}: S_j \rightarrow S_i$ are affine whenever $i \geq i_0$ (for some $i_0 \in I$).

A presentation of a scheme X as $\cup X_l^\alpha$, where $X_l^\alpha, 1 \leq l \leq n$ (it will be convenient for us to use this numbering convention throughout the paper), are pairwise disjoint locally closed subschemes of X and each X_l^α is open in

$\cup_{i \geq l} X_i^\alpha$, will be called a *stratification* of X .⁵ The corresponding embeddings $X_i^\alpha \rightarrow X$ will be denoted by j_i^α . We say that this stratification (α) is regular whenever all X_i^α are regular.

The symbol X_{red} will denote the reduced scheme associated to a scheme X . Note that X and X_{red} are “equivalent from the motivic point of view” (see Theorem 1.2.1(II.5) below); so one may assume that all the schemes we consider are also reduced (and consider all the Cartesian diagrams of this paper in the category of reduced schemes).

1.2 On K -motives over a base

For any nice scheme S consider the symmetric motivic ring spectrum $\mathrm{KGl}'_S \in \mathrm{Obj} \, SH(S)$ defined in §13.3.1 of [CiD12] (that relies on the main result of [RSO10]); this spectrum is a “minor modification” of Voevodsky’s KGl_S . Following §13.3.2 of *ibid.*, we take the category $\mathcal{DK}_{\mathbb{Z}}(S)$ of modules over KGl'_S ; here we consider KGl'_S -modules in $SH(S)$ endowed with the (“projective”) structure of a Quillen model category (see §5.3.35 of *ibid.*), and define $\mathcal{DK}_{\mathbb{Z}}(S)$ as the homotopy category of the model category obtained.

We will not give detailed proofs of (most of) the statements below since very similar statements were already proved in *ibid.* (along with other papers of Cisinski and Deglise).

Theorem 1.2.1. Let X, Y be any (nice) schemes, $f: Y \rightarrow X$ be a (separated) scheme morphism, $i \in \mathbb{Z}$.

I. Then the following statements are valid.

1. The category $\mathcal{DK}_{\mathbb{Z}}(X)$ is a tensor triangulated category; its unit object will be denoted by \mathbb{Z}_X . $\mathcal{DK}_{\mathbb{Z}}(X)$ is closed with respect to all (small) coproducts, and the tensor product respects them.
2. We have exact functors $f^*: \mathcal{DK}_{\mathbb{Z}}(X) \rightarrow \mathcal{DK}_{\mathbb{Z}}(Y)$ and $f_*: \mathcal{DK}_{\mathbb{Z}}(Y) \rightarrow \mathcal{DK}_{\mathbb{Z}}(X)$; f^* is left adjoint to f_* .

Any of them (when f varies) yields a 2-functor from the category of (nice) schemes with separated morphisms to the 2-category of triangulated categories.

3. If f is of finite type, then we also have adjoint functors

$$f_!: \mathcal{DK}_{\mathbb{Z}}(Y) \rightleftarrows \mathcal{DK}_{\mathbb{Z}}(X) : f^!$$

⁵This somewhat weak notion of a stratification was used in some previous papers of the first author.

Similarly, these two types of functors yield 2-functors from the category of (nice) schemes with separated finite type morphisms to the 2-category of triangulated categories.

4. For a Cartesian square of (separated) morphisms

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow g' & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array} \quad (1.1)$$

such that g and g' are of finite type we have $g^!f_* \cong f'_*g'^!$ and $g'_!f'^* \cong f^*g_!$.

5. f^* is symmetric monoidal; $f^*(\mathbb{Z}_X) = \mathbb{Z}_Y$.
6. $f_* \cong f_!$ if f is proper; $f^! = f^*$ if f is an open immersion.
7. If $i: Z \rightarrow X$ is a closed immersion, $U = X \setminus Z$, $j: U \rightarrow X$ is the complementary open immersion, then the motivic image functors yield a *gluing datum* for $\mathcal{DK}(-)$ in the sense of §1.4.3 of [BBD82]; cf. Proposition 1.1.2(10) of [Bon14]. In particular, for any $M \in \text{Obj } \mathcal{DK}_{\mathbb{Z}}(X)$ the pairs of morphisms

$$j_!j^*(M) \rightarrow M \rightarrow i_*i^*(M) \quad (1.2)$$

and

$$i_*i^!M \rightarrow M \rightarrow j_*j^*M \quad (1.3)$$

can be (uniquely and) functorially completed to distinguished triangles. Moreover, $i^*j_! = 0$, $i^!j_* = 0$, and the adjunction transformations $i^*i_* \rightarrow 1_{\mathcal{DK}_{\mathbb{Z}}(Z)} \rightarrow i^!i_!$ and $j^*j_* \rightarrow 1_{\mathcal{DK}_{\mathbb{Z}}(U)} \rightarrow j^!j_!$ are isomorphisms.

8. $f^! \cong f^*$ if f is smooth (of finite type).
9. If $i: S' \rightarrow S$ is a closed immersion of regular (nice) schemes then $\mathbb{Z}_{S'} \cong i^!(\mathbb{Z}_S)$.
10. If X, Y are regular, and \mathcal{O}_Y is a free \mathcal{O}_X -module of finite rank d , then for any $M \in \text{Obj } \mathcal{DK}_{\mathbb{Z}}(X)$ there exists a morphism $v \in \mathcal{DK}_{\mathbb{Z}}(X)(f_*f^*(M), M)$ such that its composition with the unit morphism $M \rightarrow f_*f^*(M)$ equals $d \text{id}_M$.
11. The full subcategory $\mathcal{DK}_{\mathbb{Z}}^c(X) \subset \mathcal{DK}(X)$ of compact objects is triangulated. $\mathcal{DK}_{\mathbb{Z}}^c(X)$ is generated by $g_!(\mathbb{Z}_{X'})$ for $g: X' \rightarrow X$ running through all smooth (finite type) morphisms.

12. f^* preserves the compactness of objects; this is also true for $f_!$ if f is of finite type.
13. f^* and f_* commute with arbitrary small coproducts; the same is true for $f_!$ and $f^!$ if f is of finite type.
14. Let a scheme S be the limit of an essentially affine (see §1.1) filtering projective system of schemes S_i for $i \in I$ (certainly, we assume that all of these schemes are nice) . Denote the corresponding transition morphisms $S_j \rightarrow S_i$ (resp. $S \rightarrow S_i$) by g_{ji} (resp. by h_i). Then $\mathcal{DK}_{\mathbb{Z}}^c(S)$ is isomorphic to the 2-colimit of the categories $\mathcal{DK}_{\mathbb{Z}}^c(S_i)$; in this isomorphism the corresponding connecting functors are given by g_{ji}^* and by h_i^* , respectively.

Furthermore, for any $i_0 \in I$, $M \in \text{Obj } \mathcal{DK}_{\mathbb{Z}}^c(S_{i_0})$, and $N \in \text{Obj } \mathcal{DK}_{\mathbb{Z}}(S_{i_0})$, the natural map

$$\varinjlim_{i \geq i_0} \mathcal{DK}_{\mathbb{Z}}(S_i)(g_{i i_0}^*(M), g_{i i_0}^*(N)) \rightarrow \mathcal{DK}_{\mathbb{Z}}(S)(h_{i_0}^*(M), h_{i_0}^*(N))$$

is an isomorphism.

15. We have $\mathcal{DK}_{\mathbb{Z}}(\mathbb{Z}_X[i], \mathbb{Z}_X) \cong KH_i(X)$; these isomorphisms (when X varies) are compatible with the motivic functors of the type f^* .

II. Assume in addition that X and Y are Λ -nice. For any S that is also Λ -nice we define the category $\mathcal{DK}(S)$ as $\mathcal{DK}_{\mathbb{Z}}(S)[\mathcal{S}^{-1}]$ (see Proposition 1.1.1); the localization functor $\mathcal{DK}_{\mathbb{Z}}(S) \rightarrow \mathcal{DK}(S)$ will be denoted by l_S .

1. $\mathcal{DK}(X)$ is a tensor triangulated category; the object $\mathbf{1}_X = l_X(\mathbb{Z}_X)$ is a unit one for this tensor product.
Furthermore, $\mathcal{DK}(X)$ is closed with respect to all (small) coproducts.
2. We have $\mathcal{DK}(\mathbf{1}_X[i], \mathbf{1}_X) \cong \mathcal{K}_i(X)$; these isomorphisms (when X varies) are compatible with the motivic functors of the type f^* . In particular, if X is regular and $i < 0$ then $\mathbf{1}_X[i] \perp \mathbf{1}_X$.
3. The natural analogues of assertions I.2–14 for the categories $\mathcal{DK}(-)$ along with their subcategories $\mathcal{DK}^c(-)$ of compact objects (and for Λ -nice schemes) are also valid.
4. If f is of finite type, then f_* and $f^!$ respect the compactness of objects (also; cf. assertion I.12).

5. If f is a finite universal homeomorphism, then f^* , f_* , $f^!$, and $f_!$ are equivalences of categories. Moreover, $f^!(\mathbf{1}_Y) \cong f^*(\mathbf{1}_Y) = \mathbf{1}_X$ and $f_*(\mathbf{1}_X) = f_!(\mathbf{1}_X) \cong \mathbf{1}_Y$.
6. If X is of finite type over a field, then $\mathcal{DK}^c(X)$ (as a triangulated category) is generated by $\{p_*(\mathbf{1}_P)\}$, where $p: P \rightarrow X$ runs through all projective morphisms such that P is regular.

In particular, if X is the spectrum of a perfect field itself, then we consider (all) smooth projective p here.

Moreover, if $\Lambda = \mathbb{Q}$ then it suffices (in the first of these statements) to assume that X is of finite type over an excellent noetherian scheme of dimension at most 3.

Proof. I. Corollary 13.3.3 of [CiD12] states that $\mathcal{DK}_{\mathbb{Z}}(-)$ is a *motivic triangulated category* (see Definition 2.4.45 of *ibid.*). Moreover, it is also oriented (see Remark 13.2.2 and Example 12.2.3(3) of *ibid.*). Hence Theorem 2.4.50 of *ibid.* (along with Definitions 2.4.45 and 1.1.21 of *ibid.*) implies our assertions 1–7. Moreover, we also obtain that assertion 8 is fulfilled “up to Tate twists” (cf. Remark 1.2.2(8) below). Since $\mathcal{DK}_{\mathbb{Z}}(-)$ is “periodic” (see (K4) in §13.2.1 of [CiD12]), the Tate twists are automorphisms of $\mathcal{DK}_{\mathbb{Z}}(-)$; this finishes the proof of this assertion.

Assertion 9 follows easily from Theorem 13.6.3 of *ibid.*

Assertion 10 can be easily established similarly to Theorem 14.3.3 of [CiD12], using Proposition 13.7.6 of *ibid.*

To prove assertion 11 we note that $g_!$ is left adjoint to g^* whenever $g: Z \rightarrow X$ is a smooth morphism. Hence the definition of $\mathcal{DK}_{\mathbb{Z}}(X)$ (see the adjunction in [CiD12, §13.3.2] and the definition of $SH(X)$) yields that all objects of the form $g_!(\mathbb{Z}_Z)$ (for a finite type g) are compact, and they Hom-generate $\mathcal{DK}_{\mathbb{Z}}(X)$ (cf. Remark 4.4 of [CiD15]). Hence the triangulated subcategory of $\mathcal{DK}_{\mathbb{Z}}(X)$ that is generated by all $g_!(\mathbb{Z}_Z)$ consists of compact objects. Lastly, it contains all compact objects of $\mathcal{DK}_{\mathbb{Z}}(X)$ by Lemma 4.4.5 of [Nee01] (cf. also Lemma A.2.10 of [Kel14]).

I.12. f^* respects the compactness of objects according to assertions I.4 and I.11. The same is true for $f_!$ (if f is of finite type) by Remark 1.2.2(3) below.

By Proposition 1.3.20 of [CiD12], these facts imply that f_* and $f^!$ (for a finite type f) respect coproducts. To conclude the proof of assertion 13 it remains to note that f^* and $f_!$ (if f is of finite type) respect coproducts since they possess right adjoints.

Next, assertion 14 can be established similarly to Proposition 4.3 of [CiD15] (see also Proposition 2.7 of *ibid.*).

15. The adjunction used in §13.3.2 of [CiD12] yields that $\mathcal{DK}_{\mathbb{Z}}(\mathbb{Z}_X[i], \mathbb{Z}_X) \cong SH(X)(\Sigma_X^\infty(X_+), \mathbf{KGl}'_X)$. It remains to apply Theorem 2.20 of [Cis13].

II. Assertions 1–3 easily follow from the corresponding statements in part I of our theorem if we combine assertions I.11, 13 with Proposition 1.1.1.

Assertion 4 can be proved similarly to Corollary 6.2.14 of [CiD16]; see Remark 3.2.6 below for more detail.

The first part of assertion 6 can be proved similarly to Proposition 7.2 of [CiD15], whereas in the $\Lambda = \mathbb{Q}$ -case one should combine Corollary 4.4.3 of [CiD12] with Theorem 1.2.5 of [Tem17]; cf. §2.4 of [BoD17]. □

Remark 1.2.2. $\mathcal{DK}(-)$ is the main motivic category of this paper.

We will need the following observations related to it below.

1. For a finite type (separated) $f: Y \rightarrow X$ we set $\mathcal{MK}_X^{BM}(Y) = f_!(\mathbf{1}_Y)$ (this is a certain *Borel–Moore* motif of Y ; cf. [BoD17] and §I.IV.2.4 of [Lev98]). For $g: X' \rightarrow X$ being a morphism of (Λ -nice) schemes (that is separated but not necessarily of finite type) one easily sees that $g^*(\mathcal{MK}_X^{BM}(Y)) \cong \mathcal{MK}_{X'}^{BM}(Y \times_X X')$.

Next, for $U \subset Y$ being an open subscheme, $Z = Y \setminus U$, the distinguished triangle (1.2) easily yields the natural distinguished triangle

$$\mathcal{MK}_X^{BM}(U) \rightarrow \mathcal{MK}_X^{BM}(Y) \rightarrow \mathcal{MK}_X^{BM}(Z). \quad (1.4)$$

(cf. §1.3.8(BM3) of [BoD17]). Certainly, this triangle yields the corresponding long exact sequence for any cohomology theory H defined on $\mathcal{DK}(X)$. We define $H_i^{BM}(Y)$ as $H(\mathcal{MK}_X^{BM}(Y)[-i])$ (so, we “lift” the index i and “change the sign of the degree in the usual way”; the reason for doing this is that we want this notation to be compatible with the usual one for K -theory).

2. More generally, let Y_l^α be the components of some stratification α of Y_{red} . Then combining obvious induction with (1.2) and Theorem 1.2.1(II.5) we obtain that any $M \in \text{Obj } \mathcal{DK}(Y)$ belongs to the extension-closure of $\{\bar{j}_{l,i} \bar{j}_i^*(M)\}$, where $\bar{j}_i: Y_l^\alpha \rightarrow Y$ are the corresponding morphisms. It easily follows that $\mathcal{MK}_X^{BM}(Y)$ belongs to the extension-closure of $\{\mathcal{MK}_X^{BM}(Y_l^\alpha)\}$. This observation is especially useful for us if α is a regular stratification.
3. Certainly, we also have distinguished triangles similar to (1.4) in $\mathcal{DK}_{\mathbb{Z}}(X)$ (for any nice X). Hence $\mathcal{DK}_{\mathbb{Z}}^c(X)$ contains $g_!(\mathbb{Z}_Z)$ for $g: Z \rightarrow X$ being an arbitrary finite type morphism. This allows to conclude the proof of Theorem 1.2.1(I.12).

4. Now we apply part 1 of this remark to (our version of) K -theory; this corresponds to $H: M \mapsto \mathcal{DK}(X)(M, \mathbf{1}_X)$. We fix a quasi-projective Z/X and choose a certain smooth Y/X containing it as a closed subscheme, $U = Y \setminus Z$. Then we obtain a long exact sequence

$$\cdots \rightarrow \mathcal{K}_i^{BM,X}(Z) \rightarrow \mathcal{K}_i(Y) \rightarrow \mathcal{K}_i(U) \rightarrow \cdots \quad (1.5)$$

here we use the fact that $g_!$ is left adjoint to g^* if g is smooth, and consider the cohomology theory represented by $\mathbf{1}_X$.

Now let $g: X' \rightarrow X$ be a morphism of schemes (that is separated and not necessarily of finite type). Then, considering the cohomology theory represented by $g_*(\mathbf{1}_{X'})$ and applying the adjunction $g^* \dashv g_*$, we get a long exact sequence

$$\begin{array}{c} \cdots \rightarrow \mathcal{K}_{i+1}(Y \times_X X') \rightarrow \mathcal{K}_{i+1}(U \times_X X') \rightarrow \\ \searrow \hspace{10em} \swarrow \\ \mathcal{K}_i^{BM,X'}(Z \times_X X') \longrightarrow \mathcal{K}_i(Y \times_X X') \rightarrow \mathcal{K}_i(U \times_X X') \\ \downarrow \cong \\ \mathcal{DK}(X)(\mathcal{MK}_X^{BM}(Z), g_*(\mathbf{1}_{X'})[-i]) \end{array} \quad (1.6)$$

It certainly follows that $\mathcal{K}_i^{BM,X'}(Z \times_X X')$ vanishes for all $i < -n$, where n is a fixed integer, if and only if $\mathcal{K}_i(Y \times_X X') \cong \mathcal{K}_i(U \times_X X')$ for $i < -n$ and $\mathcal{K}_{-n}(Y \times_X X')$ surjects onto $\mathcal{K}_{-n}(U \times_X X')$.

5. We suspect that $\mathcal{K}_i^{BM,X}(Z)$ is naturally isomorphic to the \mathcal{K} -theory of Y with the support on Z ; possibly we will treat this question in a subsequent paper (perhaps, using the results of [PPR12] or Theorem 1.18 of [Nav16]). At least, part 4 of this remark yields the following: $\mathcal{K}_i^{BM,X'}(Z \times_X X')$ vanishes for all $i \leq -n$ if and only if the same property is fulfilled for the \mathcal{K} -theory of $Y \times_X X'$ with the support on $Z \times_X X'$.
6. We will also need a certain (Verdier) dual to part 2 of this remark. Let Y_l^α be the components of some stratification α of Y_{red} . Then (1.3) combined with Theorem 1.2.1(II.5) yields that any $M \in \text{Obj } \mathcal{DK}(Y)$ belongs to the extension-closure of $\{\bar{j}_{l,*}\bar{j}_l^!(M)\}$ (for the morphisms $\bar{j}_l: Y_l^\alpha \rightarrow Y$).

Now assume that Y and all Y_l^α are regular. Then parts (I.9,II.3) of the theorem yield that $\overline{j}_l^!(\mathbf{1}_Y) \cong \mathbf{1}_{Y_l^\alpha}$ for any l . Hence $f_*(\mathbf{1}_Y)$ (for any separated $f: Y \rightarrow X$) belongs to the extension-closure of $\{f_{l,*}(\mathbf{1}_{Y_l^\alpha})\}$, where $f_l = f \circ \overline{j}_l$. It certainly follows that $f_{1,*}(\mathbf{1}_{Y_1^\alpha})$ belongs to the extension-closure of $\{f_l(\mathbf{1}_Y)\} \cup \{(f_{l,*}(\mathbf{1}_{Y_l^\alpha}) : l \geq 2)[1]$.

Lastly, assume that f is of finite type. Then for any $N \in \text{Obj } \mathcal{DK}(Y)$ the object $f^!(N)$ belongs to the extension-closure of $\overline{j}_{l,*}f_l^!(N)$. Now take $N = \mathbf{1}_X$ and assume that all Y_l^α are regular and quasi-projective over X . Then combining parts I.8, I.9, and II.3 of Theorem 1.2.1 we obtain that $f_l^!(N) = f^!(\mathbf{1}_X) \cong \mathbf{1}_{Y_l^\alpha}$. Thus $f^!(\mathbf{1}_X)$ belongs to extension-closure of $\{\overline{j}_{l,*}(\mathbf{1}_{Y_l^\alpha})\}$.

7. Below we will need only those properties of $\mathcal{DK}(-)$ that are listed in our Theorem. Thus one may consider it as a list of “axioms” for a system of triangulated categories.

In particular, the authors do not claim that all possible constructions of the categories $\mathcal{DK}(-)$ possessing these properties are isomorphic. Moreover, one can probably consider the following generalization of our setting: for R being an arbitrary torsion-free coefficient ring one may define $\mathcal{DK}(S)$ as the homotopy category of the category of modules over $\text{KGl}'_S \otimes R$ (in $SH(S)$). Yet such a generalization will not affect our main results significantly.

8. We will not need use the tensor structure much in this paper. Yet we note that for Beilinson motives (i.e., for Voevodsky motives with rational coefficients that were the central subject of [CiD12], [Heb11], and [Bon14], and will also be considered in §4 below) there is a certain particular case of tensor products that is very important (this is also the case for *cdh*-motives considered in [CiD15] and [BoI15]).

In K -motives we have $f_l(\mathbf{1}_{\mathbb{P}^1(X)}) \cong \mathbf{1}_X \oplus \mathbf{1}_X$ for any (R)-nice X . Yet for Beilinson motives (as well as for R -linear *cdh*-motives over characteristic p nice schemes, where R is unital ring such that p is invertible in R whenever it is positive) we have $f_l(\mathbf{1}_{\mathbb{P}^1(X)}) \cong \mathbf{1}_X \oplus \mathbf{1}_X\langle -1 \rangle$ for a certain \otimes -invertible object $\mathbf{1}_X\langle -1 \rangle$ (that is often denoted by $\mathbf{1}_X(-1)[-2]$; it is not isomorphic to $\mathbf{1}_X$) instead (so, Beilinson motives are not “periodic”). Tensor products by $\mathbf{1}_X\langle -1 \rangle$ and by its tensor powers (including negative ones; these product functors are called Tate twists) commute with all the motivic image functors, whereas the corresponding “axioms” of Beilinson motives differ from their K -analogues in part I.8 and I.9 of

Theorem 1.2.1 by certain (locally constant) Tate twists; see the beginning of §3.4 below for more detail.

1.3 Weight structures: reminder

Definition 1.3.1. I. For a triangulated category $\underline{\mathcal{C}}$, a pair of classes

$$\underline{\mathcal{C}}_{w \leq 0}, \underline{\mathcal{C}}_{w \geq 0} \subset \text{Obj } \underline{\mathcal{C}}$$

will be said to define a *weight structure* w for $\underline{\mathcal{C}}$ if they satisfy the following conditions:

(i) $\underline{\mathcal{C}}_{w \geq 0}$ and $\underline{\mathcal{C}}_{w \leq 0}$ are Karoubi-closed in $\underline{\mathcal{C}}$ (i.e., contain all $\underline{\mathcal{C}}$ -retracts of their objects).

(ii) **Semi-invariance with respect to translations:**

$$\underline{\mathcal{C}}_{w \leq 0} \subset \underline{\mathcal{C}}_{w \leq 0}[1], \quad \underline{\mathcal{C}}_{w \geq 0}[1] \subset \underline{\mathcal{C}}_{w \geq 0}.$$

(iii) **Orthogonality:**

$$\underline{\mathcal{C}}_{w \leq 0} \perp \underline{\mathcal{C}}_{w \geq 0}[1].$$

(iv) **Weight decompositions:** for any $M \in \text{Obj } \underline{\mathcal{C}}$ there exists a distinguished triangle

$$B \rightarrow M \rightarrow A \xrightarrow{f} B[1] \tag{1.7}$$

such that $A \in \underline{\mathcal{C}}_{w \geq 0}[1]$, $B \in \underline{\mathcal{C}}_{w \leq 0}$.

II. The full category $\underline{H}w \subset \underline{\mathcal{C}}$ whose object class is $\underline{\mathcal{C}}_{w=0} = \underline{\mathcal{C}}_{w \geq 0} \cap \underline{\mathcal{C}}_{w \leq 0}$ will be called the *heart* of w .

III. $\underline{\mathcal{C}}_{w \geq i}$ (resp. $\underline{\mathcal{C}}_{w \leq i}$, resp. $\underline{\mathcal{C}}_{w=i}$) will denote $\underline{\mathcal{C}}_{w \geq 0}[i]$ (resp. $\underline{\mathcal{C}}_{w \leq 0}[i]$, resp. $\underline{\mathcal{C}}_{w=0}[i]$).

IV. We will call $\underline{\mathcal{C}}^b = (\cup_{i \in \mathbb{Z}} \underline{\mathcal{C}}_{w \leq i}) \cap (\cup_{i \in \mathbb{Z}} \underline{\mathcal{C}}_{w \geq i})$ the class of *bounded* objects of $\underline{\mathcal{C}}$. We will say that w is bounded if $\underline{\mathcal{C}}^b = \text{Obj } \underline{\mathcal{C}}$.

Besides, we will say that elements of $\cup_{i \in \mathbb{Z}} \underline{\mathcal{C}}_{w \geq i}$ are *bounded below*.

V. Let $\underline{\mathcal{C}}$ and $\underline{\mathcal{C}}'$ be triangulated categories endowed with weight structures w and w' , respectively; let $F: \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}'$ be an exact functor.

The functor F will be said to be *left weight-exact* (with respect to w, w') if it maps $\underline{\mathcal{C}}_{w \leq 0}$ into $\underline{\mathcal{C}}'_{w' \leq 0}$; it will be called *right weight-exact* if it maps $\underline{\mathcal{C}}_{w \geq 0}$ to $\underline{\mathcal{C}}'_{w' \geq 0}$. F is called *weight-exact* if it is both left and right weight-exact.

Remark 1.3.2. 1. A weight decomposition (of any $M \in \text{Obj } \underline{\mathcal{C}}$) is (almost) never canonical; still (any choice of) a pair (B, A) coming from (1.7) will be often denoted by $(w_{\leq 0}M, w_{\geq 1}M)$. More generally, for any $m \in \mathbb{Z}$ shifting a weight decomposition of $M[-m]$ by $[m]$ we obtain a distinguished triangle $w_{\leq m}M \rightarrow M \rightarrow w_{\geq m+1}M$ with some $w_{\geq m+1}M \in \underline{\mathcal{C}}_{w \geq m+1}$, $w_{\leq m}M \in \underline{\mathcal{C}}_{w \leq m}$; we will call it an m -weight decomposition of M .

2. A simple (and yet useful) example of a weight structure comes from the stupid filtration on $K(B)$ (for an arbitrary additive category B ; see §1.1). We take $K(B)_{w \leq 0} = K(B)^{\geq 0}$ (in the notation described above), and take $K(B)_{w \geq 0}$ being the similarly defined $K(B)^{\leq 0}$. We call this weight structure the *stupid* one; see Remark 1.2.3(1) of [BoS18] for more detail.

3. In the current paper we use the “homological convention” for weight structures; it was previously used in [Heb11], [Bon14], and in successive papers, whereas in [Bon10a] and in [Bon10b] the “cohomological convention” was used. In the latter convention the roles of $\underline{\mathcal{C}}_{w \leq 0}$ and $\underline{\mathcal{C}}_{w \geq 0}$ are interchanged, i.e., one considers $\underline{\mathcal{C}}^{w \leq 0} = \underline{\mathcal{C}}_{w \geq 0}$ and $\underline{\mathcal{C}}^{w \geq 0} = \underline{\mathcal{C}}_{w \leq 0}$. For example, a complex $M \in \text{Obj } K(B)$ whose only non-zero term is the fifth one has weight -5 (with respect to the stupid weight structure) in the homological convention, and has weight 5 in the cohomological convention. Thus the conventions differ by “signs of weights”.

Now we recall some basic properties of weight structures.

Proposition 1.3.3. Let $\underline{\mathcal{C}}$ be a triangulated category endowed with a weight structure w , $M \in \text{Obj } \underline{\mathcal{C}}$, $i, j \in \mathbb{Z}$. Then the following statements are valid.

1. The axiomatics of weight structures is self-dual, i.e., for $\underline{D} = \underline{\mathcal{C}}^{op}$ (so $\text{Obj } \underline{\mathcal{C}} = \text{Obj } \underline{D}$) there exists the (opposite) weight structure w' for which $\underline{D}_{w' \leq 0} = \underline{\mathcal{C}}_{w \geq 0}$ and $\underline{D}_{w' \geq 0} = \underline{\mathcal{C}}_{w \leq 0}$.
2. $\underline{\mathcal{C}}_{w \leq i}$, $\underline{\mathcal{C}}_{w \geq i}$, and $\underline{\mathcal{C}}_{w=i}$ are Karoubi-closed and extension-closed in $\underline{\mathcal{C}}$ (and so, additive).
3. Assume that $j < i$. Then for any choices of weight decompositions corresponding to the rows of the square

$$\begin{array}{ccc} w_{\leq j}M & \longrightarrow & M \\ \downarrow c_{ji} & & \downarrow \text{id}_M \\ w_{\leq i}M & \longrightarrow & M \end{array}$$

there exists a unique morphism c_{ji} making it commutative. Moreover, $\text{Cone}(c_{ji}) \in \underline{\mathcal{C}}_{[j+1, i]}$.

4. $\underline{C}_{w \geq i} = (\underline{C}_{w \leq i-1})^\perp$ and $\underline{C}_{w \leq i} = {}^\perp \underline{C}_{w \geq i+1}$.
5. The class $\underline{C}_{[i,j]}$ equals the extension-closure of $\cup_{i \leq k \leq j} \underline{C}_{w=k}$
6. If $A \rightarrow B \rightarrow C \rightarrow A[1]$ is a \underline{C} -distinguished triangle and $A, C \in \underline{C}_{w=0}$, then $B \cong A \oplus C$.
7. If $A \rightarrow C \rightarrow B$ is a \underline{C} -distinguished triangle then for any fixed $w_{\leq i-1}A, w_{\leq i-1}B$ there exists a weight decomposition of C such that $w_{\leq i-1}C$ is an extension of $w_{\leq i-1}B$ by $w_{\leq i-1}A$.
8. Let \underline{D} be a triangulated category endowed with a weight structure v ; let $F: \underline{C} \rightleftarrows \underline{D}: G$ be exact adjoint functors. Then F is left weight-exact if and only if G is right weight-exact.
9. Assume that M belongs to the envelope (see §1.1) of some class of $M_j \in \text{Obj } \underline{C}$ (for $j \in J$); we fix a choice of $w_{\leq i-1}M_j$ (for these objects). Then $M \in \underline{C}_{w \geq i}$ if and only if $w_{\leq i-1}M_j \perp M$.
In particular, if $J = J_1 \cup J_2$ such that $M_j \in \underline{C}_{w \geq i}$ for any $j \in J_1$ and $M_j \in \underline{C}_{w \leq i-1}$ for any $j \in J_2$, then it suffices to check whether $M_j \perp M$ for all $j \in J_2$.

Proof. Assertions 1–6 are contained in Theorem 2.2.1 of [Bon10b] (whereas their proofs relied on [Bon10a]) and assertion 7 is a part of Lemma 1.5.4 of [Bon10a] (pay attention to Remark 1.3.2(3)!).

Assertion 8 is just Proposition 1.2.3(9) of [Bon14].

It remains to verify assertion 9. Obviously, the “in particular” part of the assertion is just a particular case of the general statement preceding it; so we prove the latter. Furthermore, the orthogonality axiom of weight structures yields that $N \perp M$ if $N \in \underline{C}_{w \leq i-1}$ and $M \in \underline{C}_{w \geq i}$; so the “only if” part of the statement is clear.

We prove the converse implication. We start with the following easy observation: for any choice of an $i-1$ -weight decomposition triangle $w_{\leq i-1}M \rightarrow M \rightarrow w_{\geq i}M$ we have $M \in \underline{C}_{w \geq i}$ if and only if $w_{\leq i-1}M \perp M$. Indeed, the latter condition yields that M is a retract of $w_{\geq i}M$ (whereas the converse implication is immediate).

Hence it suffices to verify that any object M in the envelope of $\{M_j\}$ possesses a shifted weight decomposition (as above) such that $w_{\leq i-1}M$ belongs to the envelope of $\{w_{\leq i-1}M_j\}$. To this end it certainly suffices to combine assertion 7 with the following statement: for A , some fixed $w_{\leq i-1}A$ and C being a retract of A there exists a weight decomposition of C such that $w_{\leq i-1}C$ is a retract of $w_{\leq i-1}A$. The latter result can easily be established using the

corresponding arguments from the proof of Lemma 5.2.1 of *ibid.* (at least, in the case where \underline{C} is Karoubian; cf. the remark below).

□

Remark 1.3.4. Whereas the case of a Karoubian \underline{C} is certainly sufficient for the purposes of the current paper, we note that there exists an alternative (and more elegant) proof of assertion 9. It relies on the properties of the so-called virtual t -truncations of cohomological functors (see §2.3 of [Bon10b] and Appendix A.3 of [Bon14]). We consider the functor $F = \tau_{\geq 1-i}(\underline{C}(-, M)) : \underline{C}^{op} \rightarrow Ab$; note that it is cohomological (also). If $w_{\leq i-1} M_j \perp M$ (for all j) then $F(M_j) = 0$. Hence in this case we have $F(M) = \{0\}$. It remains to note that the latter implies that $M \in \underline{C}_{w \geq i}$.

In §2.3 and §4 below we will need some properties of weight structures “extended” from subcategories of compact objects.

Proposition 1.3.5. Let $\underline{C}' \subset \underline{C}$ be triangulated categories such that \underline{C} contains all small coproducts of its objects, \underline{C}' is essentially small, and the objects of \underline{C}' are compact in \underline{C} . Let w' be a bounded weight structure on \underline{C}' . Then the following statements are valid.

1. The sets $\underline{C}_{w \geq 0} = \underline{C}'_{w' \leq -1}{}^{\perp_{\underline{C}}}$ and $\underline{C}_{w \leq 0} = {}^{\perp_{\underline{C}}}(\underline{C}_{w \geq 0}[1])$ yield a weight structure on \underline{C} .
2. \underline{Hw} equals the \underline{C} -Karoubi-closure of the category of all \underline{C} -coproducts of objects of \underline{Hw}' .
3. Assume that $\underline{C}'_{w' \leq 0}$ is the envelope of some set $\{C_i\}$ of its objects. Then $\underline{C}_{w \geq 0} = \{C_i[-1]\}^{\perp_{\underline{C}}}$.
4. The embedding $\underline{C}' \rightarrow \underline{C}$ is weight-exact (with respect to w and w').
5. Let v be a weight structure on a triangulated category \underline{D} ; let $F : \underline{C} \rightleftarrows \underline{D} : G$ be adjoint exact functors. Then the following statements are equivalent:
 - (a) $F(\underline{C}'_{w' \leq 0}) \subset \underline{D}_{v \leq 0}$.
 - (b) G is right weight-exact.
 - (c) F is left weight-exact.
6. For (\underline{C}, w) and (\underline{D}, v) as above and any exact $G : \underline{D} \rightarrow \underline{C}$ that commutes with (small) coproducts we have the following: G is left weight-exact whenever $G(\underline{D}'_{v' \leq 0}) \subset \underline{C}_{w \leq 0}$.

7. For any $M_i \in \text{Obj } \underline{\mathcal{C}}$ (for i running through some index set) we have $\coprod M_i \in \underline{\mathcal{C}}_{w \geq 0}$ (resp. $\coprod M_i \in \underline{\mathcal{C}}_{w \leq 0}$) if and only if $M_i \in \underline{\mathcal{C}}_{w \geq 0}$ (resp. $M_i \in \underline{\mathcal{C}}_{w \leq 0}$) for all i .

Proof. We will apply the following obvious observation and denote it by the symbol (*): for any objects $M_i \in \text{Obj } \underline{\mathcal{C}}$ and $M \subset \text{Obj } \underline{\mathcal{C}}$ the class $\{M_i\}^\perp \subset M^\perp$ whenever M lies in the envelope of $\{M_i\}$.

The orthogonality axiom of weight structures immediately gives $\underline{\mathcal{C}}'_{w'=0} \perp_{\underline{\mathcal{C}}} \underline{\mathcal{C}}'_{w'=0}[i]$ for all $i > 0$; in other words, the class $\underline{\mathcal{C}}'_{w'=0}$ is *negative* in $\underline{\mathcal{C}}$ (see Remark 2.2.2(3) of [BoS17]). Since the elements of $\underline{\mathcal{C}}'_{w'=0}$ are compact in $\underline{\mathcal{C}}$, $\underline{\mathcal{C}}'_{w'=0}$ is also *class-negative* in $\underline{\mathcal{C}}$ (see Definition 1.2.2(10) and Remark 2.3.2(2) of *ibid.*). Applying Corollary 2.3.1(1) of *ibid.* we obtain the existence of a class $L \subset \text{Obj } \underline{\mathcal{C}}$ such that the couple $(L, (\cup_{i < 0} \underline{\mathcal{C}}'_{w'=0}[i])^\perp_{\underline{\mathcal{C}}})$ is a weight structure (on $\underline{\mathcal{C}}$). Next, combining the observation (*) with Proposition 1.3.3(2,5) we obtain that the class $\underline{\mathcal{C}}'_{w' \leq -1} \perp_{\underline{\mathcal{C}}}$ equals $(\cup_{i < 0} \underline{\mathcal{C}}'_{w'=0}[i])^\perp_{\underline{\mathcal{C}}}$. Invoking Proposition 1.3.3(4) we conclude that the couple described in assertion 1 is a weight structure indeed.

Applying Corollary 2.3.1(1) of *ibid.* once again we also obtain assertion 2.

Assertion 3 immediately follows from our observation (*).

Assertion 4 is an immediate consequence of the description of w along with Proposition 1.3.3(4).

Given assertion 1, assertion 5 is provided by Remark 2.1.5(3) of [Bon16b].

Assertion 6 is given by Corollary 2.3.1(1) of [BoS17] also.

Lastly, *loc. cit.* also says that w is *smashing* in the sense of Definition 1.2.2(7) of *ibid.*, and the latter fact immediately implies assertion 7. □

Remark 1.3.6. 1. Theorem 5 of [Pau12] states that one can obtain a weight structure on $\underline{\mathcal{C}}$ “starting from” the right envelope of any set of objects of $\underline{\mathcal{C}}$ (instead of $\underline{\mathcal{C}}_{w \leq -1}$ in assertion 1); cf. also Theorem 4.4.3 of [Bon16b](1) for a more general statement. Yet we do not need these more general existence results in the current paper.

2. One can obtain several more properties of weight structures obtained this way by combining *loc. cit.* with Proposition 2.5.1 of *ibid.*

1.4 On weight complexes and weight spectral sequences

Now we recall some of the properties of weight complexes and weight spectral sequences.⁶ The only place preceding §4 where they will be needed is Remark

⁶The term “weight complex” originates from [GiS96]; yet the functor of Gillet and Soulé was (essentially) extended to Voevodsky motives over a field only in [Bon09], whereas the

2.3.3; so that the reader mostly interested in sections 2–3 may ignore the current section.

Proposition 1.4.1. Let $\underline{\mathcal{C}}$ be a triangulated category endowed with a weight structure w , $M \in \text{Obj } \underline{\mathcal{C}}$, $n \in \mathbb{Z}$. Then the following statements are valid.

1. For all $i \in \mathbb{Z}$ fix some choices of $w_{\leq i}M$ and denote $\text{Cone}(c_{i-1,i})[-i]$ (see Proposition 1.3.3(3) by M^{-i} . Define (a choice of) the weight complex $t(M)$ for M as the complex whose terms are M^j (for $j \in \mathbb{Z}$) and the connecting morphisms are given as the corresponding compositions $M^j \rightarrow (w_{\leq -j-1}M)[j+1] \rightarrow M^{j+1}$.

Then $t(M)$ is a complex indeed (i.e., the square of the boundary is zero); all M^i belong to $\underline{\mathcal{C}}_{w=0}$ (so, we are able to consider $t(M)$ as an object of $K(\underline{Hw})$).

2. M determines $t(M)$ up to a homotopy equivalence.
3. If M is bounded below, then $M \in \underline{\mathcal{C}}_{w \geq -n}$ if and only if $t(M) \in K(\underline{Hw})^{\leq n}$.
4. If M belongs to the envelope of certain $M_i \in \text{Obj } \underline{\mathcal{C}}$ then $t(M)$ belongs to the $K(\underline{Hw})$ -envelope of $t(M_i)$.
5. Let $d_i: M_{\leq i-1} \rightarrow M_{\leq i}$ for $i \in \mathbb{Z}$ be a chain of $\underline{\mathcal{C}}$ -morphisms such that $\text{Cone}(d_i) \in \underline{\mathcal{C}}_{w=i}$ for all i , $M_i = 0$ for $i \ll 0$, and $M_i \cong M$ (with d_{i+1} being isomorphisms) for $i \gg 0$. Then $M_{\leq i}$ give certain choices of $w_{\leq i}M$ and d_i yield the corresponding $c_{i-1,i}$.

Proof. All of these statements except the last one are essentially contained in Theorem 3.3.1 of [Bon10a] (yet pay attention to Remark 1.3.2(3)!). Assertion 5 is precisely Theorem 2.2.1(14) of [Bon10b]. \square

Now we recall some basics on (general) weight spectral sequences. It will be more convenient for us in this paper to consider them for homological functors (including perverse étale homology of motives); certainly, dualization is not a problem (cf. §2.4 of [Bon10a]).

Proposition 1.4.2. Let \underline{A} be an abelian category; $H: \underline{\mathcal{C}} \rightarrow \underline{A}$ be any functor.

current general definition was given in [Bon10a]; cf. also §2.4 of [Bon16b]. Furthermore, our definition of weight spectral sequences (essentially and) vastly generalizes the one of Deligne; see Remark 2.4.3 of [Bon10a], §3.6 of [Bon14], and Proposition 4.3.1 below.

I. For any $m \in \mathbb{Z}$ the object $(W_m H)(M) = \text{Im}(H(w_{\leq m} M) \rightarrow H(M))$ does not depend on the choice of $w_{\leq m} M$; moreover, it is \underline{C} -functorial in M .

II. Now let $H: \underline{C} \rightarrow \underline{A}$ be a homological functor; for any $r \in \mathbb{Z}$ denote $H \circ [r]$ by H_r .

Then the following statements are valid.

1. There exists a (*weight*) spectral sequence $T = T_w(H, M)$ with $E_1^{pq} = H_q(M^p)$ such that the differentials for $E_1 T_w(H, M)$ come from $t(M)$. It converges to $H_{p+q}(M)$ if M is bounded.

2. $T_w(H, M)$ is \underline{C} -functorial in M (and does not depend on any choices) starting from E_2 .

3. If M is bounded, then the step of filtration given by $(E_\infty^{l, m-l} : l \geq k)$ on $H_m(M)$ (for some $k, m \in \mathbb{Z}$) equals $(W_{-k} H_m)(M)$.

Proof. Immediate from Proposition 2.1.2 and Theorem 2.3.2 of [Bon10a] (cf. also Proposition 1.3.2 of [Bon15]). \square

Corollary 1.4.3. Let $\underline{C}, \underline{A}, H, M$ be as in (part II of) the proposition, $n \in \mathbb{Z}$, and consider the following conditions:

1. $E_2^{pq} T_w(H, M) \neq 0$ for some $q \in \mathbb{Z}$ and $p > n$.
2. $(W_{-n-1} H_q)(M) \neq 0$ for some $q \in \mathbb{Z}$.

Then the following statements are valid.

1. If condition 1 or 2 is fulfilled then $M \notin \underline{C}_{w \geq -n}$.
2. Assume that M is bounded below. Then condition 2 implies condition 1.

Proof. 1. It suffices to note that one can take $w_{\leq i} M = 0$ for all $i < -n$ when computing weight filtrations and weight spectral sequences (using weight complexes).

2. Assume that $M \in \underline{C}_{w \geq k}$ for some $k \in \mathbb{Z}$. We should check that $(W_{-n-1} H_q)(M) = 0$ for all $q \in \mathbb{Z}$ and $p > n$ whenever $E_2^{pq} T_w(H, M) = 0$ (for $p > n$).

Fix some choices of $w_{\leq i} M$ for all $i \in \mathbb{Z}$; we take $w_{\leq i} M = 0$ for $i < k$. According to Proposition 1.3.3(3) the corresponding object $M' = w_{\leq -n} M$ belongs to $\underline{C}_{[k, -n]}$ and $w_{\leq j} M$ yield some choices of $w_{\leq j} M'$ for $j \leq -n$. Hence the corresponding choice of $t(M')$ is the stupid truncation of $t(M)$ in degrees $\geq n$.

Now we consider the weight spectral sequence $T_w(H, M)$. According to Proposition 1.4.2, the vanishing of $E_2^{pq} T_w(H, M) = E_2^{pq} T_w(H, M')$ for $p > n$ yields that $(W_{-n-1} H_q)(M') = \text{Im}(H(W_{-n-1} M') \rightarrow H(M')) = 0$ (for all q). Thus $(W_{-n-1} H_q)(M) = 0$ also. \square

2 On Chow-weight structures for K -motives; relating the “motivic length” with negative K -groups

In this section we introduce the Chow weight structures for $\mathcal{DK}(-)$ and relate them to (negative) \mathcal{K} -groups.

In §2.1 we introduce and study the Chow weight structures on the subcategories $\mathcal{DK}^c(-) \subset \mathcal{DK}(-)$ of compact objects. Our exposition closely follows the arguments of [BoI15].

In §2.2 we use a simple calculation to establish the relation between the weights of $f_*(\mathbf{1}_Y)$ (for a separated finite type $f: Y \rightarrow X$) to the (negative degree, Borel–Moore) \mathcal{K} -groups of Y -schemes. We prove several equivalent conditions for $f_*(\mathbf{1}_Y) \in \mathcal{DK}(X)_{w_{Chow} \geq -n}$.

In §2.3 we use Proposition 1.3.5 to extend the Chow weight structures and their properties from $\mathcal{DK}^c(-)$ to $\mathcal{DK}(-)$. This allows us to lift the finiteness of type restriction on f that was imposed in the criteria of Theorem 2.2.1.

2.1 On the Chow weight structure for compact K -motives

Similarly to [Bon14] and [BoI15], the properties of K -motives listed in Theorem 1.2.1 yield the existence of certain *Chow* weight structures for $\mathcal{DK}^c(-) \subset \mathcal{DK}(-)$. We will start with the properties of the “compact” Chow weight structure.

Sometimes we will need the definition of “global” Chow motives over a scheme.

Definition 2.1.1. For a scheme S we define the category $Chow(S)$ of Chow motives over S as the Karoubi-closure of $\{\mathcal{MK}_S^{BM}(P)\}$ in $\mathcal{DK}(S)$; here P runs through all finite type regular schemes that are projective over S .

Note that the “Beilinson-motivic” versions of Chow motives (as defined here) have played a crucial role in [Heb11] and in [Bon14, §2.1–2.2]; cf. also §2.3 of [BoI15].

Theorem 2.1.2. Let X be an Λ -nice scheme. Then the following statements are valid.

I. There exists a bounded weight structure $w_{Chow}^e = w_{Chow}^e(X)$ for $\mathcal{DK}^c(X)$ that possesses the following descriptions.

1. $\mathcal{DK}^c(X)_{w_{Chow}^e \leq 0}$ is the right envelope of $\{\mathcal{MK}_X^{BM}(P)\}$, $\mathcal{DK}^c(X)_{w_{Chow}^e \geq 0}$ is the left envelope of $p_*(\mathbf{1}_P)$ for $p: P \rightarrow X$ running through all compositions of a smooth projective morphism with a finite universal homeomorphism whose domain is regular and with an immersion.

2. Moreover, $\mathcal{DK}^c(X)_{w_{\mathcal{C}how}^c \leq 0}$ is the right envelope of $\{\mathcal{MK}_X^{BM}(T)\}$ for T running through all finite type X -schemes; $\mathcal{DK}^c(X)_{w_{\mathcal{C}how}^c \geq 0}$ is the left envelope of $\{t_*(\mathbf{1}_T)\}$ for $t: T \rightarrow X$ running through all finite type morphisms with **regular** domains.
3. $\mathcal{DK}^c(X)_{w_{\mathcal{C}how}^c \geq 0}$ is the right envelope of $\{\mathcal{MK}_X^{BM}(T)\}$ for T running through all quasi-projective X -schemes; $\mathcal{DK}^c(X)_{w_{\mathcal{C}how}^c \leq 0}$ is the left envelope of $t_*(\mathbf{1}_T)$ for $t: T \rightarrow X$ running through all quasi-projective morphisms with regular domains.
4. If X of finite type over a field, then $\mathcal{DK}^c(X)_{w_{\mathcal{C}how}^c \leq 0}$ is the right envelope of $\text{Obj } \mathcal{C}how(S)$, $\mathcal{DK}^c(X)_{w_{\mathcal{C}how}^c \geq 0}$ is the left envelope of $\text{Obj } \mathcal{C}how(S)$.

II. Let $f: Y \rightarrow X$ be a (separated) scheme morphism. Then the following statements are valid for the ‘‘compact versions’’ of the motivic functors.

1. $\mathbf{1}_X \in \mathcal{DK}^c(X)_{w_{\mathcal{C}how}^c \leq 0}$.
2. If the reduced scheme X_{red} associated to X is regular then $\mathbf{1}_X \in \mathcal{DK}^c(X)_{w_{\mathcal{C}how}^c = 0}$.
3. f^* is left weight-exact.
4. If f is of finite type then $f^!$ and f_* are right weight-exact, and $f_!$ is left weight-exact.
5. If f is proper then $f_* = f_!$ is also weight-exact.
6. Moreover, $f^* \cong f^!$ is weight-exact if f is either smooth or a finite universal homeomorphism.
7. Moreover, f^* is weight-exact if it is the (inverse) limit of an essentially affine system such that the transition morphisms are compositions of smooth morphisms and finite universal homeomorphisms.
8. Let \mathcal{X} denote the set of (Zariski) points of X ; for a $K \in \mathcal{X}$ the corresponding morphism $K \rightarrow X$ is denoted by j_K . Then $M \in \mathcal{DK}^c(X)_{w_{\mathcal{C}how}^c \leq 0}$ if and only if for any $K \in \mathcal{X}$ we have $j_K^*(M) \in \mathcal{DK}^c(K)_{w_{\mathcal{C}how}^c \leq 0}$.
9. Define $j_K^!$ using the ‘‘standard method’’ (cf. §2.2.12 of [BBD82]) as follows: we decompose j_K as $K \xrightarrow{j_K^0} \overline{K} \xrightarrow{i} X$ (\overline{K} is the closure of K in X) and set $j_K^! = i^! \circ j_K^{0*}$.

Then $M \in \mathcal{DK}^c(X)_{w_{\mathcal{C}how}^c \geq 0}$ if and only if for any $K \in \mathcal{X}$ (for \mathcal{X} as above) we have $j_K^!(M) \in \mathcal{DK}^c(K)_{w_{\mathcal{C}how}^c \geq 0}$.

Proof. I. The methods applied in the proof of Theorem 2.1.3, Theorem 2.2.1(2d) (along with Remark 2.2.2(1)), and Proposition 2.3.2 (along with Remark 2.3.3(1)) of [BoI15] can be carried over to our context (of Λ -linear K -motives over Λ -nice schemes) without any difficulty (if we apply Theorem 1.2.1).

II. Once again, the arguments used in (§2.2 of) [BoI15] yield all of the statements without any difficulty. □

Remark 2.1.3. 1. The arguments used in [BoI15] for the proof of the corresponding analogue of (the existence statement in) Theorem 2.1.2 were not “very explicit”; in particular, they do not bound the weights of $\mathbf{1}_X$ from below (for a non-regular X). In the next section we will describe certain more explicit arguments (that can also be applied to $DM(-)$).

2. The corresponding envelopes in [BoI15] “differed by stabilization” via $\otimes \mathbf{1}_X \langle i \rangle (= \mathbf{1}_X \langle -1 \rangle^{\otimes -i}$ for $i \in \mathbb{Z}$; see Remark 1.2.2(8)). The natural analogues of the properties of w_{Chow}^c described in this section are also valid for Beilinson motives (over arbitrary nice base schemes) since they possess all the properties needed to prove them.

3. One of the main benefits of weight structures is that they relate the objects of \underline{C} to the “more simple” objects of the corresponding \underline{Hw} (via weight complexes and weight spectral sequences). So, it certainly makes sense to describe $\underline{Hw}_{Chow(X)}^c$ “more explicitly” (note that this will also yield a description of the whole $\underline{Hw}_{Chow(X)}$; see Proposition 2.3.1 below and Proposition 1.3.5(2)). Now, Theorem 1.2.1(II.6) (combined with Theorem 4.3.2(II) of [Bon10a]) implies that $\underline{Hw}_{Chow(X)}^c = Chow(X)$ (see Definition 2.1.1) whenever either X is of finite type over a field or $\Lambda = \mathbb{Q}$ and X is of finite type over an excellent noetherian scheme of dimension at most 3. Now, one can “compute” morphism groups between two objects of the type $\mathcal{MK}_X^{BM}(P)$ where P is regular and projective over X using the method of the proof of [Bon14, Lemma 1.1.4.(I.1)]. Computing the composition of morphisms operation for these motives seems to be the most difficult problem here; yet one may probably solve it by applying the arguments of [Jin16].

2.2 Motivic weight bounds in the terms of negative \mathcal{K} -groups: the finite type version

Now we combine the previous theorem with Proposition 1.3.5(3) (and apply these statements to certain motives of X -schemes). We start from the case

of X -schemes of finite type.

Theorem 2.2.1. Let $f: Y \rightarrow X$ be a (separated) finite type morphism of Λ -nice schemes, $n \geq 0$. Then the following conditions are equivalent.

1. $f_*(\mathbf{1}_Y) \in \mathcal{DK}^c(X)_{w_{\mathcal{C}how} \geq -n}$.
2. For any (separated) finite type morphism $P \rightarrow X$ the groups $\mathcal{K}_i^{BM,Y}(P \times_X Y)$ (see Remark 1.2.2(1, 4)) vanish for $i < -n$.
3. For any morphism $P \rightarrow X$ of finite type with regular affine domain the groups $\mathcal{K}_i^{BM,Y}(P \times_X Y)$ vanish for $i < -n$.
4. For any morphism $P \rightarrow X$ being the composition of a smooth projective morphism with a finite universal homeomorphism whose domain is regular and with an immersion the groups $\mathcal{K}_i^{BM,Y}(P \times_X Y)$ vanish for $i < -n$.
5. For any smooth P/X and any open embedding $j: U \rightarrow P$ we have the following: $\mathcal{K}_i(P \times_X Y) = \{0\}$ for $i < -n$ and $\mathcal{K}_{-n}(P \times_X Y)$ surjects onto $\mathcal{K}_{-n}(U \times_X Y)$.

Proof. If condition 1 is fulfilled then we certainly have $\mathcal{DK}^c(X)_{w_{\mathcal{C}how} \leq n-1} \perp f_*(\mathbf{1}_Y)$. Hence for any separated finite type P/X and $i < n$ we have $\mathcal{MK}_X^{BM}(P)[i] \perp f_*(\mathbf{1}_Y)$ (see Theorem 2.1.2(I.2)). It remains to apply the isomorphism $\mathcal{K}_i^{BM}(Z \times_X Y) \cong \mathcal{DK}(X)(\mathcal{MK}_X^{BM}(Z), f_*(\mathbf{1}_Y[-i]))$ (mentioned in (1.6)) to deduce condition 2.

Certainly, condition 2 yields conditions 4 and 3. Conversely, condition 3 implies condition 2 according to Remark 1.2.2(2).

Similarly to the implication 1 \implies 2, condition 4 yields that $\mathcal{MK}_X^{BM}(P)[i] \perp f_*(\mathbf{1}_Y)$ for any P of the corresponding type. Hence it implies condition 1 by Theorem 2.1.2(I.2) (along with Proposition 1.3.3(4)).

Lastly, since $\mathcal{K}_i(P \times_X Y) \cong \mathcal{K}_i^{BM,Y}(P \times_X Y)$ (basically by definition), condition 2 implies condition 5 according to Remark 1.2.2(4). Next, condition 5 certainly yields that the corresponding facts are true for P being smooth quasi-projective over X . Hence the remark cited yields condition 1 if we combine it with Theorem 2.1.2(I.3) (along with Proposition 1.3.3(4)).

□

Remark 2.2.2. 1. The “initial” case $X = Y$, as well as the cases where X is the spectrum of a field (that may be perfect or just equal to \mathbb{C}) and (the “final” case) $X = \text{Spec } \Lambda$ in the theorem (already) appear to be quite interesting. Note that by the virtue of the results of the

following subsection we can consider arbitrary separated morphisms (not necessarily of finite type) of Λ -nice schemes in this remark.

Now we define $c(Y/X)$ as the minimal $n \in \mathbb{Z}$ such that $f_*(\mathbf{1}_Y) \in \mathcal{DK}(X)_{w_{Chow} \geq -n}$. Theorem 2.1.2 (along with Proposition 2.3.1 in the case when f is not of finite type, and with the following part of this remark) yields the following properties of this characteristic: $c(Y/X) \geq 0$; $c(Y/X) = 0$ if X_{red} is regular; $c(Y'/X) \leq c(Y/X)$ if Y' is smooth over Y ; $c(Y/X) \geq c(Y/S)$ if X is separated over S .

2. Applying the argument used in the proof of our theorem for $P = X$ one also obtains $f_*(\mathbf{1}_Y) \notin \mathcal{DK}(X)_{w_{Chow} \geq 1}$ (since $\mathcal{K}_0(Y) \neq \{0\}$).

Yet note that criteria quite similar to the ones above can be proved for the question whether $\text{Cone}(f'_*(\mathbf{1}_{Y'}) \rightarrow f_*(\mathbf{1}_Y))$ belongs to $\mathcal{DK}(X)_{w_{Chow} \geq -n}$ (where the connecting morphism is induced by a factorization of f through some finite type $f': Y' \rightarrow X$). This (more general) formulation can also be interesting for $n < 0$.

3. Certainly, conditions 4 and 3 can also be reformulated similarly to condition 5.

Note also that one may replace the Borel–Moore K -theory in these conditions by the corresponding K -theory with support; see Remark 1.2.2(5).

4. Below we will show that in condition 2 one can restrict himself to a finite set of “test schemes” P/X . Now we will only make two simple remarks related to this claim.

Firstly, if X is of finite type over a field, it suffices to consider P running through regular projective X -schemes (see Theorem 2.1.2(I.4)). Note also that in the case where X is the spectrum of a field itself, one may pass to its perfect closure (using Theorem 2.1.2(II.7)). After that one can consider P being smooth over (the new) X , and so we have $\mathcal{K}_i^{BM,Y}(P \times_X Y) = \mathcal{K}_i(P \times_X Y)$.

On the other hand, (1.4) implies that one can “cut test schemes into pieces”. Moreover, one can pass to direct limits here; this yields certain “Borel–Moore stalks” (that are closely related to certain coniveau spectral sequences); see [BoD17].

2.3 On “ \mathcal{K} -weight bounds” for X -schemes that are not of finite type

Now we prove the “non-compact” version of Theorem 2.1.2. It easily implies that Theorem 2.2.1 can be extended to Y being not necessarily of finite type over X .

Proposition 2.3.1. For any (Λ -nice) scheme S let $w_{Chow} = w_{Chow}(S)$ denote the “extension” of $w_{Chow}^c(S)$ from $\mathcal{DK}^c(S)$ to $\mathcal{DK}(S)$ constructed via the method of Proposition 1.3.5(1). Let $f: Y \rightarrow X$ be a (separated) scheme morphism, $M \in \text{Obj } \mathcal{DK}(X)$. Then the following statements are valid.

1. $\mathbf{1}_X \in \mathcal{DK}(X)_{w_{Chow} \leq 0}$.
2. If X_{red} (see the end of §1.1) is regular then $\mathbf{1}_X \in \mathcal{DK}(X)_{w_{Chow} = 0}$.
3. f_* is right weight-exact; f^* is left weight-exact.
4. If f is of finite type then $f^!$ is right weight-exact and $f_!$ is left weight-exact.
5. If f is proper then $f_* = f_!$ is also weight-exact.
6. Moreover, $f^* \cong f^!$ is weight-exact if f is either smooth or a finite universal homeomorphism.
7. $f_*(\mathbf{1}_Y)$ is bounded below.
8. Let α be a stratification of X . Then for the corresponding $j_l: X_l^\alpha \rightarrow X$ we have the following statements: $M \in \mathcal{DK}(X)_{w_{Chow} \leq 0}$ if and only if $j_l^*(M) \in \mathcal{DK}(X_l^\alpha)_{w_{Chow} \leq 0}$ for all l ; $M \in \mathcal{DK}(X)_{w_{Chow} \geq 0}$ if and only if $j_l^!(M) \in \mathcal{DK}(X_l^\alpha)_{w_{Chow} \geq 0}$ for all l .
9. Let $x_i: X_i \rightarrow X$ be an open cover of X . Then $M \in \mathcal{DK}(X)_{w_{Chow} \leq 0}$ (resp. $M \in \mathcal{DK}(X)_{w_{Chow} \geq 0}$) if and only if $x_i^*(M)$ belongs to $\mathcal{DK}(X_i)_{w_{Chow} \leq 0}$ (resp. to $\mathcal{DK}(X_i)_{w_{Chow} \geq 0}$) for all i .
10. f^* is weight-exact also in the case where f is the (projective) limit of an essentially affine system such that the transition morphisms are compositions of smooth morphisms and finite universal homeomorphisms.
11. Let \mathcal{X} and $j_K^!$ (for $K \in \mathcal{X}$) be as in Theorem 2.1.2(II.8). Then $M \in \mathcal{DK}(X)_{w_{Chow} \geq 0}$ if and only if for any $K \in \mathcal{X}$ we have $j_K^!(M) \in \mathcal{DK}(K)_{w_{Chow} \geq 0}$.

Proof. All of these statements are easy consequences of Theorem 2.1.2 along with Proposition 1.3.5.

In particular, Proposition 1.3.5(4) (combined with Theorem 2.1.2) yields assertions 1 and 2.

Next, combining Proposition 1.3.5(5) with the adjunctions $f^* \dashv f_*$ and $f_! \dashv f^!$ (for f being a finite type morphism) we obtain the proof of assertions 3 and 4. Assertions 5 and 6 follow from the previous two immediately when we take into account the isomorphisms of functors mentioned in their formulations.

Next, $\mathbf{1}_Y$ is bounded below (in $\mathcal{DK}(Y)$) since $w_{Chow}^c(Y)$ is bounded. Hence $f_*(\mathbf{1}_Y)$ is bounded below also, and we obtain assertion 7. Next, $\mathbf{1}_Y$ is bounded below (in $\mathcal{DK}(Y)$) since $w_{Chow}^c(Y)$ is bounded. Hence $f_*(\mathbf{1}_Y)$ is bounded below also, and we obtain assertion 7.

The “only if” statements in assertion 8 are immediate from assertions 3 and 4, respectively.

To obtain the converse implications one should recall that the classes $\mathcal{DK}(X)_{w_{Chow} \leq 0}$ and $\mathcal{DK}(X)_{w_{Chow} \geq 0}$ are extension-closed. Thus it suffices to recall that M belongs to the extension-closure of $\{j_{!}j_l^*(M)\}$ and also to the extension-closure of $\{j_{!*}j_l^!(M)\}$ (see Remark 1.2.2(2, 6)) and apply the remaining statements in assertions 3 and 4.

9. Note that $j_i^! = j_i^*$ and take a stratification of X such that each of X_i^α lies in one of the subschemes X_i . Then the assertion follows from the previous one easily.

To prove assertion 10 it certainly suffices to verify that f^* is right weight-exact. So, for any $M \in \mathcal{DK}(X)_{w_{Chow} \geq 0}$, any finite type Y'/Y and $i < 0$ we should check that $\mathcal{MK}_Y^{BM}(Y')[i] \perp f^*(M)$ (see Proposition 1.3.5(3) and Theorem 2.1.2(I.2)). Now, we can assume that Y' is defined “at a finite level” (i.e., for $f = \varinjlim f_i: Y_i \rightarrow X$ there exists i_0 such that $Y' = Y \times_{Y_{i_0}} Y'_0$ for some finite type Y'_0/Y_{i_0} ; see Theorem 8.8.2(ii) of [EGA4III]). Hence for the corresponding $h_{i_0}: Y \rightarrow X_{i_0}$ we have $\mathcal{MK}_Y^{BM}(Y') \cong h_{i_0}^*(\mathcal{MK}_{Y_{i_0}}^{BM} Y'_0)$ (see Remark 1.2.2(1)); for the transition morphisms $g_{ii_0}: Y_i \rightarrow Y_{i_0}$ we also have $\mathcal{DK}(Y)(\mathcal{MK}_Y^{BM}(Y')[i], f^*(X)) \cong \varinjlim_{i \geq i_0} DM_{Y_i}(\mathcal{MK}_{Y_i}^{BM} Y_i \times_{Y_{i_0}} Y'_0, g_{ii_0}^*(M))$ (see Theorem 1.2.1(I.14, II.3)). Now, the weight-exactness of g_{ii_0} implies the vanishing of (all terms of) this limit; this concludes the proof of the assertion.

The “only if” part of assertion 11 follows immediately from the previous assertion (along with assertion 4). The converse implication can be proved similarly to its “compact motivic” version (i.e., to the corresponding “half” of Proposition 2.2.3 of [Bo15]); note that Theorem 1.2.1(I.14, II.3) contains more information than its analogue used in *ibid.*

□

Corollary 2.3.2. Theorem 2.2.1 is also valid for Y being not of finite type (and separated) over X .

Proof. The proof of Theorem 2.2.1 can be extended to this context without any difficulty if we apply Proposition 2.3.1 and use Proposition 1.3.5(3) instead of Proposition 1.3.3(4). □

Remark 2.3.3. 1. Now we recall that $g_*(\mathbf{1}_X)$ is bounded below; hence it belongs to $\mathcal{DK}(X)_{w_{Chow} \geq -n}$ if and only if $t(g_*(\mathbf{1}_X)) \in K(\underline{Hw}_{Chow}(X))^{\leq n}$ (see Proposition 1.4.1(3)). This criterion does not seem to be very “practical” in general (especially if g is not of finite type). Yet certainly there are some cases where $t(g_*(\mathbf{1}_X))$ can be computed “explicitly”. Possibly, we will consider this question in a subsequent paper.

2. Once again (following Remark 1.2.2(8)) we recall that in the cases $\Lambda = \mathbb{Q}$ and $\Lambda = \mathbb{Z}[\frac{1}{p}]$ (for any $p \in \mathbb{Z}$) there exist “reasonable” Λ -linear “Voevodsky-type” motivic categories $DM(-)$ over arbitrary Λ -nice schemes. All of the statements of this section easily carry over to these categories; the main distinction is that one should “take Tate twists into account” (see the aforementioned remark once again), whereas in Theorem 1.2.1(II.2) one should take the corresponding Λ -linear motivic cohomology instead of (Λ -linear homotopy invariant) K -theory. We did not treat this setting in detail since this sort of motivic cohomology (of singular schemes) seems to be “less popular” than K -theory, and we didn’t want to restrict ourselves to two possibilities for Λ only.

3 “Studying weights” using (general) Gabber’s resolution of singularities results

The main goal of this section is to establish some weight bounds on objects of the type $f_*(\mathbf{1}_Y)$ and $\mathcal{MK}_X^{BM}(Y)$ for Y being (usually) of finite type over X . We also put objects of this type in certain “explicit” envelopes (that yield related envelope statements for some choices of m -weight decompositions for these motives, where m runs through integers). We also obtain similar results for other “motivic categories” possessing properties similar to that of K -motives. Since we want our results to be as general as possible, we have to invoke some definitions and results of O. Gabber (as described in [ILO14]). Note also that most of the formulations of this section are motivated by the example described in Remark 3.4.3(4) below.

In §3.1 we introduce a certain modification of Gabber’s dimension functions; the reason for this is that the properties of the corresponding function $\delta = \delta^B$ is “more satisfactory” for our purposes than the properties of Krull dimension.

In §3.2 we formulate and prove the aforementioned envelope statements for K -motives.

In §3.3 we combine these results with the ones of the previous section to obtain a collection of weight bounds and the corresponding bounds on “the negativity” of certain (Borel–Moore) \mathcal{K} -groups. In particular, we prove that K_{-s} is “supported in codimension s ” (for any $s \geq 0$).

In §3.4 we extend the results of §3.2 to (any system of) motivic triangulated categories $\mathcal{D}(-)$ satisfying certain (“axiomatic”) assumptions. In particular, this yields a collection of results on étale sheaves that appear to be new.

3.1 On dimension functions

Since we want our results to be valid for a wide range of schemes, we will need a certain “substitute” of the Krull dimension function $\dim(-)$.⁷ So we need certain “pseudo-dimension functions” that we define in terms of Gabber’s dimension functions (as introduced in §XIV.2 of [ILO14] and applied to motives in [BoD17]).

Definition 3.1.1. Let B be a Λ -nice scheme; denote by B^k its irreducible components whose dimensions are d_k . Let y be the spectrum of a field that is essentially of finite type over B (i.e., it is the generic point of an irreducible scheme of finite type over B) and b be its image in B .

1. Throughout this section we will say that a scheme is a B -scheme only if it is separated and of finite type over B .
2. If $b \in B^k$ then we define $\delta^k(y) = d_k - \text{codim}_{B^k} b + \text{tr. deg. } k(y)/k(b)$, where $k(y)$ and $k(b)$ are the corresponding residue fields.

For a general y we define $\delta(y)$ as the minimum over all B^k containing b of the numbers $\delta^k(y)$.

3. For Y being an S_0 -scheme we define $\delta^B(Y) = \delta(Y)$ as the maximum over points of Y of $\delta(y)$.

⁷The reason is that we want some notion of dimension that would satisfy the following property: if U is open dense in X then its “dimension” should equal the “dimension” of X .

The results of [ILO14, §XIV.2] easily yield that the function δ satisfies the following properties.

Proposition 3.1.2. Let X and U be B -schemes.

1. $\delta(X)$ is a finite integer that is not smaller than $\dim(X)$.
2. $\delta(U) \leq \delta(X) + d$ whenever there exists a B -morphism $u: U \rightarrow X$ generically of dimension at most d . Moreover, we have an equality here whenever $d = 0$ and u is dominant, and a strict inequality if the image of u is nowhere dense.
3. Furthermore, if $c > 0$, $U \subset X$, and any irreducible component of U is of codimension at least c in some irreducible component of X (containing it) then $\delta(U) \leq \delta(X) - c$.

Proof. For any irreducible component B^k of B we combine Proposition XIV.2.2.2 of [ILO14] (cf. also the proof of Corollary XIV.2.2.4 of loc. cit.) with Corollary XIV.2.5.2 of ibid. to obtain that the restriction of δ^k to the points of any B^k -scheme Y yields a dimension function on it (in the sense of Definition XIV.2.1.10 of ibid.). It follows immediately that δ^k satisfy the obvious analogues of our assertions. Combining these statements for all k we easily obtain the result. □

Remark 3.1.3. 1. If B is of finite type over a field or over $\text{Spec } \mathbb{Z}$ (more generally, it suffices to assume that B is a Jacobson scheme all of whose components are equicodimensional; see Proposition 10.6.1 of [EGA4III]) then our method yields a function δ such that $\delta(Y) = \dim(Y)$ for any Y that is of finite type over B . Thus the reader satisfied with this restricted setting may replace all δ -dimensions in this section by Krull dimensions. Moreover, some of the arguments may be simplified. Another easier case is the one of irreducible B . In both of these cases our δ is a “true” dimension function, i.e., $\delta(X) - \delta(Z) = \text{codim}_X(Z)$ for $Z \subset X$ being any irreducible B -schemes.

2. Recall that any B -scheme Y possesses a B -compactification \bar{Y} by Nagata’s theorem (i.e., Y is open dense in \bar{Y} and \bar{Y} is proper over B ; see Theorem 4.1 of [Con07]). Since $\delta(\bar{Y}) = \delta(Y)$, we obtain $\delta(Y) \geq \dim(\bar{Y})$.

Note also that the (Krull) dimensions of all possible B -compactifications of Y are equal.

3. It is easily seen for B' being a B -scheme that the values of the similarly defined function $\delta^{B'}$ is not greater than the (corresponding) ones of δ .

4. Actually, in all the statements involving δ before Corollary 3.2.7 we will use Proposition 3.1.2 as a certain “axiomatics” of δ . In particular, if (a possibly, reducible) B admits a dimension function in the sense of [ILO14, Definition XIV.2.1.10] then one may use the corresponding “extension” of this function to B -schemes (see Corollary XIV.2.5.2 of *ibid.*) instead of δ . Recall also that any Λ -nice S possesses an open cover by subschemes admitting dimension functions; see Corollary XIV.2.2.4 of [ILO14]. This statement may be used to enhance slightly Theorem 3.3.1(I.2) below, and to extend the results of [BoD17]. It may also make sense to combine the results of this section with usage of somewhat more general (“generalized”) dimension functions described in Definition 4.2.1 of [BoS18].
5. Actually, in the most of the formulations below taking $B = X$ seems to be “optimal”.

3.2 On the “motivic Gabber’s lemma” and its applications

The arguments of this section are mostly based on §6.2 of [CiD16]; yet our “basic resolution of singularities diagram” is the one used in [Kel14]. Another distinction from the arguments is that the usage of Verdier localizations certainly does not “give control” over those envelopes that are not shift-stable; so we apply Proposition 1.1.3 instead.

Throughout the subsection we will assume that all the schemes we consider are of finite type over some fixed (nice) B .

Definition 3.2.1. Let X be finite type separated B -scheme, $d = \delta(X)$, $r \geq 0$.

1. Given a closed embedding $i_Z: Z \rightarrow W$ of schemes of finite type over X and an open $U \subset X$ we consider the following commutative diagram:

$$\begin{array}{ccccc}
 Z & \xrightarrow{i_Z} & W & \xleftarrow{j_W} & U_W \\
 & \searrow \pi & \downarrow a & & \downarrow a_U \\
 & & X & \xleftarrow{j} & U;
 \end{array}$$

here the right-hand square is Cartesian.

We define the following object in $\mathcal{DK}(X)$:

$$\varphi_X^U(Z \rightarrow W) = \pi_* i_Z^* j_{W,*}(\mathbf{1}_{U_W}) = \pi_* i_Z^* j_{W,*} a_U^*(\mathbf{1}_U);$$

we will often omit X and U in this notation. Respectively, when using this notation we will suppose that the corresponding X, Z, W, U satisfy the aforementioned conditions. Q will usually denote $X \setminus U$ and $Q_W = W \times_X Q$; denote the embeddings $Q \rightarrow X$ and $Q_W \rightarrow W$ by i and i_W , respectively, and denote the composition $\alpha \circ i_W$ by π_Q .

In the case $W = X$, $Z = Q$ we will often replace the notation $\varphi_X^U(Q \rightarrow X)$ just by $\varphi(X) = \varphi_X^U(X)$.

2. We define the following classes of objects in $\mathcal{DK}(X)$: $N_r(X) = N_r^B(X)$ is the $\mathcal{DK}(X)$ -envelope of $\{\mathcal{MK}_X^{BM}(V)[m]\}$ for V running through regular finite type X -schemes, $m \geq 0$, and $\delta(V) + m \leq r$; $VN_r(X) = VN_r^B(X) \subset N_r(X)$ is the envelope of $\{\mathcal{MK}_X^{BM}(V)[m]\}$ with (V, m) satisfying all the conditions for $N_r(X)$ along with $\delta(V) < r$.
3. We will say that j is *r-reasonable* if the corresponding object $\varphi_X^U(X)$ belongs to $N_r(X)$. We will say that j is just *reasonable* if it is *d-reasonable*; then we will call (U, X) a *reasonable pair*.
More generally, for $c > 0$ we will say that (U, X) is *c-coreasonable* if there exists $T \subset Q$ such that $\dim X - \dim T \geq c$ and the pair $(U, X \setminus T)$ is reasonable.
4. We will say that a reduced scheme X' is *reasonable* if (V', X') is reasonable for any open dense regular $V' \subset X'$.

Remark 3.2.2. 1. We certainly have $N_{r-1}(X)[1] \subset VN_r(X) \subset N_r(X)$ (assuming that $N_{r-1}(X)$ is empty for $r = 0$).

2. Certainly, any element of $N_r(X)$ (or of $VN_r(X)$) belongs to the envelope of $\{\mathcal{MK}_X^{BM}(V_b)[m_b]\}$ for some finite set of (V_b, m_b) as above. Yet our arguments yield some more information on possible (V_b, m_b) corresponding to a reasonable embedding (see part 3 of the definition). We start explaining this from certain easy and useful arguments; see the continuation in Remark 3.3.2(3) below.

So, let $f: Y \rightarrow X$ and $g: X \rightarrow S$ be (separated) finite type morphisms of B -schemes. Then $g_!$ certainly sends $VN_r(X) \subset N_r(X)$ into $VN_r(S) \subset N_r(S)$.

Next, for a regular finite type V/X we have $f^*(\mathcal{MK}_X^{BM}(V)[m]) \cong \mathcal{MK}_Y^{BM}(V \times_X Y)[m]$. Thus for V_b being the components of some regular stratification of $(V \times_X Y)_{red}$ the object $f^*(\mathcal{MK}_X^{BM}(V)[m])$ belongs to the $\mathcal{DK}(Y)$ -envelope of $\{\mathcal{MK}_Y^{BM}(V_b)[m]\}$ (see Remark 1.2.2(2)). Hence $f^*(VN_r(X)) \subset VN_r(Y)$ and $f^*(N_r(X)) \subset N_r(Y)$ whenever f is quasi-finite.

3. Hence one may assume that all V_b corresponding to part 3 of the definition are Q -schemes (since $i_!i^*(\varphi_X^U(X)) \cong \varphi_X^U(X)$).

Now we prove a collection of properties of our notions.

Lemma 3.2.3. Adopt the notation of Definition 3.2.1 (so, $d = \delta(X)$).

1. There is a distinguished triangle $j_!(\mathbf{1}_U) \xrightarrow{h} j_*(\mathbf{1}_U) \rightarrow \varphi_X^U(X)$.

Thus if we assume that $\varphi_X^U(X)$ belongs to the envelope of $\{\mathcal{MK}_X^{BM}(V_b)[m_b]\}$ for some finite type morphisms $v_b: V_b \rightarrow X$ and $m_b \in \mathbb{Z}$, then $j_*(\mathbf{1}_U)$ belongs to the envelope of $\{\mathcal{MK}_X^{BM}(U)\} \cup \{\mathcal{MK}_X^{BM}(V_b)[m_b + 1]\}$ and $\mathcal{MK}_X^{BM}(V)$ belongs to the envelope of $\{j_*(\mathbf{1}_U)\} \cup \{\mathcal{MK}_X^{BM}(V_b)[m_b]\}$.

2. Let $g: X' \rightarrow X$ be a smooth morphism; denote by W', U', Z' the pull-backs of the corresponding schemes with respect to g . Then $g^*(\varphi_X^U(Z \rightarrow W)) \cong \varphi_{X'}^{U'}(Z' \rightarrow W')$.
3. $\varphi_X^U(Z \rightarrow W)$ is contravariantly functorial in (Z, W) , i.e., any commutative square C

$$\begin{array}{ccc} Z' & \longrightarrow & W' \\ \downarrow k_Z & & \downarrow k \\ Z & \longrightarrow & W \end{array} \quad (3.1)$$

of finite type X -schemes yields a morphism $\varphi(C): \varphi(Z \rightarrow W) \rightarrow \varphi(Z' \rightarrow W')$, and this construction respects compositions of morphisms of pairs.

4. If the morphism k in (3.1) is étale and k_Z is an isomorphism then $\varphi(C)$ is an isomorphism.
5. Let Z' be a closed subscheme of Z ; denote the embedding of $\tilde{Z} = Z \setminus Z'$ into W by \tilde{i} . Then there is a distinguished triangle $D \rightarrow \varphi(Z \rightarrow W) \rightarrow \varphi(Z' \rightarrow W) \rightarrow D[1]$ for $D = a_*\tilde{i}_! \tilde{i}^* j_{W,*}(\mathbf{1}_{U_W})$, where the second arrow is the one coming from assertion 3. Moreover, D is isomorphic to $a_*\tilde{j}_!(\varphi_{W \setminus Z'}^{U_W}(\tilde{Z} \rightarrow W \setminus Z'))$ for \tilde{j} being the embedding $W \setminus Z' \rightarrow W$ whenever U_W is disjoint from Z , whereas $D \cong (a \circ \tilde{i})_*(\mathbf{1}_{\tilde{Z}})$ whenever $\tilde{Z} \subset U_W$.
6. Assume that $W = X$ and let $m: M \rightarrow X$ be an embedding. Denote by \overline{M} the closure of M in X , and let $M \xrightarrow{\tilde{m}} \overline{M} \xrightarrow{\overline{m}} X$ be the corresponding factorization. Then $E = \pi_*\pi^*m_*(\mathbf{1}_M)$ is isomorphic to $\overline{m}_!(\varphi_{\overline{M}}^M(Z \cap \overline{M} \rightarrow \overline{M}))$. In particular, $E \cong \overline{m}_!\tilde{m}_*(\mathbf{1}_M) = m_*(\mathbf{1}_M)$ whenever $M \subset Z$.

7. Let T be a closed subscheme of Q ; denote its embedding into X by t . Then $\varphi_X^U(T \rightarrow X) \cong t_! t^*(\varphi_X^U(X))$. Moreover, for $o: O \rightarrow X$ being the embedding complementary to t there is a distinguished triangle

$$o_!(\varphi_O^{U \cap O}(Q \setminus T \rightarrow O)) \rightarrow \varphi_X^U(X) \rightarrow \varphi_X^U(T \rightarrow X) \quad (3.2)$$

8. Assume that $U = \sqcup U_b$; denote by c_b the embeddings of the X -closures $\overline{U_b}$ into X , $Q_b = \overline{U_b} \setminus U_b$. Then $\varphi_X^U(X) \cong \bigoplus_b c_{b,!}(\varphi_{\overline{U_b}}^{U_b}(Q_b \rightarrow \overline{U_b}))$.
9. Assume that a is proper, U and U_W are regular, \mathcal{O}_{U_W} is a free \mathcal{O}_U -module of dimension e . Then the morphism $\varphi(X) \rightarrow \varphi(Q_W \rightarrow W)$ (coming from the corresponding commutative diagram via assertion 3) is split injective (i.e., is a coretraction) in the localized category $\mathcal{DK}(X)[e^{-1}]$.
10. Let O_b be a Zariski cover of X . Then j is r -reasonable whenever all the embeddings $j \times_X O_b$ are.
11. Assume that j is dense, W is regular, $\delta(W) = r$, π is proper, $Z \subset Q_W$, and all closed nowhere dense reduced subschemes of W are reasonable. Then $\varphi_X^U(Z \rightarrow X) \in VN_r(X)$.
12. If j is reasonable, U is regular and dense in X , then X is reasonable.
13. For any $s > 0$ we have $N_{r-s}(X)[-s] \subset VN_{r-s+1}(X)[-s] \subset N_r(X)$.

Proof. 1. By Theorem 1.2.1(I.7) we have a distinguished triangle

$$j_! j^* j_*(\mathbf{1}_U) \rightarrow j_*(\mathbf{1}_U) \rightarrow i_* i^* j_*(\mathbf{1}_U),$$

and there is a canonical isomorphism $j^* j_*(\mathbf{1}_U) \cong \mathbf{1}_U$. It remains to note that $\varphi_X^U(X) = i_* i^* j_*(\mathbf{1}_U)$ by definition.

2. Easy, using smooth base change (Theorem 1.2.1(I.4, I.6)).
3. We have a commutative diagram

$$\begin{array}{ccccc} Z' & \xrightarrow{i'_Z} & W' & \xleftarrow{j'_W} & U'_W \\ k_Z \downarrow & & \downarrow k & & \downarrow k_U \\ Z & \xrightarrow{i_Z} & W & \xleftarrow{j_W} & U_W \\ & \searrow \pi & \downarrow a & & \downarrow a_U \\ & & X & \xleftarrow{j} & U \end{array}$$

where the right-hand squares are pullbacks. Let's start with a unit morphism of the adjunction $k_U^* \dashv k_{U,*}$:

$$\mathbf{1}_{U_W} \rightarrow k_{U,*}k_U^*(\mathbf{1}_{U_W}) = k_{U,*}(\mathbf{1}_{U'_W}).$$

Applying $j_{W,*}$, we get a morphism

$$j_{W,*}(\mathbf{1}_{U_W}) \rightarrow j_{W,*}k_{U,*}(\mathbf{1}_{U'_W}) \cong k_*j'_{W,*}(\mathbf{1}_{U'_W}).$$

Using the adjunction $k^* \dashv k_*$, we get a morphism

$$k^*j_{W,*}(\mathbf{1}_{U_W}) \rightarrow j'_{W,*}(\mathbf{1}_{U'_W}).$$

Applying $i_Z'^*$, we get

$$k_Z^*i_Z'^*j_{W,*}(\mathbf{1}_{U_W}) \cong i_Z'^*k^*j_{W,*}(\mathbf{1}_{U_W}) \rightarrow i_Z'^*j'_{W,*}(\mathbf{1}_{U'_W}).$$

Using the adjunction $k_Z^* \dashv k_{Z,*}$, we get a morphism

$$i_Z'^*j_{W,*}(\mathbf{1}_{U_W}) \rightarrow k_{Z,*}i_Z'^*j'_{W,*}(\mathbf{1}_{U'_W}).$$

Finally, applying π_* , we get

$$\varphi(Z \rightarrow W) = \pi_*i_Z'^*j_{W,*}(\mathbf{1}_{U_W}) \rightarrow (\pi \circ k_Z)_*i_Z'^*j'_{W,*}(\mathbf{1}_{U'_W}) = \varphi(Z' \rightarrow W').$$

It is clear that this construction respects composition.

4. This is the K -motivic version of Lemma 6.2.11 of [CiD16], and its proof carries over to our setting without any difficulty. Indeed, note that (in the notation of the proof of part 3) in our setting the morphism

$$k^*j_{W,*}(\mathbf{1}_{U_W}) \rightarrow j'_{W,*}(\mathbf{1}_{U'_W})$$

is an isomorphism. Then we apply $i_Z'^*$ to it, use the adjunction $k_Z^* \dashv k_{Z,*}$, and apply π_* ; thus the result is again an isomorphism.

5. Note that (in the setting of the proof of part 3) the morphisms k and k_U are identities. Therefore the morphism $\varphi(Z \rightarrow W) \rightarrow \varphi(Z' \rightarrow W)$ is obtained by applying π_* to the unit morphism $M \rightarrow k_{Z,*}k_Z^*M$, where $k_Z: Z' \rightarrow Z$ is the closed embedding, and $M = i_Z'^*j_{W,*}\mathbf{1}_{U_W}$. Now, the gluing datum of Theorem 1.2.1 (I.7) gives a distinguished triangle

$$\tilde{k}_{Z,!}\tilde{k}_Z^*M \rightarrow M \rightarrow k_{Z,*}k_Z^*M,$$

where $\tilde{k}_Z: \tilde{Z} \rightarrow Z$ is the open embedding. After applying π_* , we get the desired triangle. The ‘‘moreover’’ part of the assertion easily follows from the fact that $\tilde{i}^*j_{W,!} = 0$ (combined with assertion 1).

6. We have $m_*(\mathbf{1}_M) \cong \overline{m}_! \tilde{m}_*(\mathbf{1}_M)$. Consider the corresponding Cartesian square in the category of reduced schemes

$$\begin{array}{ccc} Z \cap \overline{M} & \xrightarrow{\overline{m}_Z} & Z \\ \downarrow \pi_{\overline{M}} & & \downarrow \pi \\ \overline{M} & \xrightarrow{\overline{m}} & X \end{array}$$

Then Theorem 1.2.1(I.4, II.3, II.5) yields $\pi^* \overline{m}_! \cong \overline{m}_{Z,!} \pi_{\overline{M}}^*$. Hence

$$E \cong \pi_* \overline{m}_{Z,!} \pi_{\overline{M}}^* \tilde{m}_*(\mathbf{1}_M) = \overline{m}_!(\varphi_{\overline{M}}^M(Z \cap \overline{M} \rightarrow \overline{M})).$$

Lastly, if $M \subset Z$, then $Z \cap \overline{M} = \overline{M}$ and $\pi_{\overline{M}} = \text{id}_{\overline{M}}$; therefore $\varphi_{\overline{M}}^M(Z \cap \overline{M} \rightarrow \overline{M}) = \tilde{m}_*(\mathbf{1}_M)$. It follows that $E \cong \overline{m}_! \tilde{m}_*(\mathbf{1}_M) = m_*(\mathbf{1}_M)$.

7. The first part of the assertion is immediate from $t_* t^* \cong t_! t^* i_* i^*$.

Now we consider the distinguished triangle $o_! o^*(\varphi_X^U(X)) \rightarrow \varphi_X^U(X) \rightarrow t_! t^*(\varphi_X^U(X))$ coming from (1.2). It yields (3.2) since $o^*(\varphi_X^U(X)) \cong \varphi_O^{U \cap O}(Q \setminus T \rightarrow O)$ according to assertion 2.

8. Easy; note that $j_*(\mathbf{1}_U) \cong \bigoplus_b j_{b,*}(\mathbf{1}_{U_b})$ (for the corresponding embeddings $j_b: U_b \rightarrow X$).

9. The argument is similar to the proof of Lemma 6.2.12 of *ibid.* We have a commutative diagram

$$\begin{array}{ccccc} Q_W & \xrightarrow{i_W} & W & \xleftarrow{j_W} & U_W \\ a_Q \downarrow & & \downarrow a & & \downarrow a_U \\ Q & \xrightarrow{i} & X & \xleftarrow{j} & U, \end{array}$$

where both squares are pullbacks. Note that $\varphi(X) = \varphi(Q \rightarrow X) = i_* i^* j_*(\mathbf{1}_U)$ and

$$\begin{aligned} \varphi(Q_W \rightarrow W) &= (a \circ i_W)_* i_W^* j_{W,*} a_U^*(\mathbf{1}_U) \\ &= (i \circ a_Q)_* i_W^* j_{W,*} a_U^*(\mathbf{1}_U) \\ &= i_* i^* a_{*,*} j_{W,*} a_U^*(\mathbf{1}_U) \\ &= i_* i^* j_* a_{U,*} a_U^*(\mathbf{1}_U). \end{aligned}$$

Here we used the proper base change formula (Theorem 1.2.1(I.4, I.6)). It is easy to see that the morphism $\varphi(Q \rightarrow X) \rightarrow \varphi(Q_W \rightarrow W)$ constructed in part 3 is obtained by applying $i_* i^* j_*$ to the unit morphism $\mathbf{1}_U \rightarrow a_{U,*} a_U^*(\mathbf{1}_U)$. Now we can apply Theorem 1.2.1(I.10).

10. We should check the reasonability of (V, X) for any open dense regular $V \subset X$. Choose a stratification α of X such that any of X_l^α lies inside one of the O_b . Now, according to Remark 1.2.2(2) it suffices to verify that $j_{l,!}j_l^*(\varphi_X^V(X \setminus V \rightarrow X)) \in VN_r(X)$ for all l . Hence it remains to combine assertion 2 with Remark 3.2.2(2).

11. Remark 3.2.2(3, 2) yields that it suffices to verify the assertion with X replaced by W ; so we assume $W = X$.

Choose a regular stratification α of X such that $X_1^\alpha = U$ and any of X_l^α for $l > 1$ either lies in Z or is disjoint from Z . Then $\delta(X_l^\alpha) < d$ for $l \geq 2$ (see Proposition 3.1.2), and $j_*(\mathbf{1}_U)$ belongs to the envelope of $\{\mathbf{1}_X\} \cup \{j_{l,*}(\mathbf{1}_{X_l^\alpha})[1] : l > 1\}$ (see Remark 1.2.2(6)). Thus it suffices to verify that the objects $\pi_*\pi^*(\mathbf{1}_X) = \mathcal{MK}_X^{BM}(Z)$ and $\pi_*\pi^*j_{l,*}(\mathbf{1}_{X_l^\alpha}[1])$ for $l > 1$ belong to $VN_r(X)$.

Now, $\mathcal{MK}_X^{BM}(Z) \in N_{r-1}(X) \subset VN_r(X)$ immediately from Remark 1.2.2(2).

Next we apply assertion 6. So we present all j_l as the compositions $X_l^\alpha \xrightarrow{\bar{j}_l} \overline{X}_l^\alpha \xrightarrow{i_l} X$, where \overline{X}_l^α are the closures of X_l^α in X . Recall that the embeddings \bar{j}_l are reasonable according to our assumptions.

If X_l^α (for $l > 1$) lies in Z then $\pi_*\pi^*j_{l,*}(\mathbf{1}_{X_l^\alpha}[1]) \cong i_{l,!}\bar{j}_{l,*}(\mathbf{1}_{X_l^\alpha}[1])$ according to (the ‘‘in particular’’ part of) assertion 6. Now, $\bar{j}_{l,*}(\mathbf{1}_{X_l^\alpha}) \in N_{r-1}(\overline{X}_l^\alpha)$ according to (the ‘‘thus’’ part of) assertion 1; hence it remains to apply Remark 3.2.2(2, 1).

Lastly, assume that X_l^α is disjoint from Z for some $l > 1$; denote $Z \cap \overline{X}_l^\alpha$ by Z_l . Then $\pi_*\pi^*j_{l,*}(\mathbf{1}_{X_l^\alpha}[1]) \cong i_{l,!}\varphi_{\overline{X}_l^\alpha}^{X_l^\alpha}(Z_l \rightarrow \overline{X}_l^\alpha)[1]$ according to assertion 6.

Since \bar{j}_l is reasonable, we have $\varphi_{\overline{X}_l^\alpha}^{X_l^\alpha}(\widetilde{X}_l^\alpha \rightarrow \overline{X}_l^\alpha) \in VN_{r-1}(\overline{X}_l^\alpha)$, where $\widetilde{X}_l^\alpha = \overline{X}_l^\alpha \setminus X_l^\alpha$. Since Z_l is a closed subscheme of \widetilde{X}_l^α , it remains to combine (the first part of) assertion 7 with Remark 3.2.2(2, 1).

12. It certainly suffices to prove the following (for any open dense regular $U \subset X$): if U is regular and $j_1: U_1 \rightarrow U$ is an open dense embedding then j is reasonable if and only if $j'_1 = j \circ j_1$ is.

Now we argue somewhat similarly to the proof of assertion 11. Choose a regular stratification α of U such that $U_1^\alpha = U_1$. Certainly, $\delta(U_l^\alpha) < d$ for $l > 1$. Denote $j \circ j_l$ by j'_l for all l . Then $j_*(\mathbf{1}_U)$ belongs to the envelope of $\{j'_{l,*}(\mathbf{1}_{U_l})\}$ and $j'_{l,*}(\mathbf{1}_{U_l})$ belongs to the envelope of $\{j_*(\mathbf{1}_U)\} \cup \{j'_{l,*}(\mathbf{1}_{U_l}) : l \geq 2\}[1]$. Hence the arguments used in the proof of assertion 11 can easily be applied to conclude the proof.

13. These inclusions easily follow the fact that $\mathcal{MK}_X^{BM}(V)[-s]$ is a retract of $\mathcal{MK}_X^{BM}(\mathbb{G}_m^s(V))$ and $\delta(\mathbb{G}_m^s(V)) = \delta(V) + s$ (where \mathbb{G}_m^s is the s th Cartesian power of the multiplicative group scheme).

□

Now we are able to prove the central (technical) statement of this section.

Proposition 3.2.4. Any reduced B -scheme is reasonable.

Proof. We will verify the reasonability of any (finite type) reduced B -scheme X of δ -dimension at most d for some $d \geq 0$. Note that the statement is trivial for $d = 0$ (since then $\dim(X) = 0$ according to Proposition 3.1.2).

Moreover, a finite type (separated) X/B , $\delta(X) = d$, is reasonable whenever it is $d + 1$ -coreasonable. Thus by an obvious induction on (c, d) our proposition reduces to the following claim (somewhat similarly to §6.2 of [CiD16] and §XIII.3 of [ILO14]): X is $c + 1$ -coreasonable for some integer c , $0 \leq c \leq d$, provided that any finite type reduced B -scheme is reasonable if its δ -dimension is less than d and is c -coreasonable if its δ -dimension equals d .

Now, it suffices to verify our claim in the case where X is irreducible. Indeed, let $X = \cup X_k$, where X_k are irreducible components of X . If all X_k are $c + 1$ -coreasonable then one can choose closed $T_k \subset X_k$ of δ -dimension less than $d - c$ and non-empty open regular $U_k \subset X_k \setminus T_k$ such that all the pairs $(U_k, X_k \setminus T_k)$ are reasonable and U_k are pairwise disjoint (in X). Then $(\sqcup U_k, X \setminus (\cup_k T_k))$ is easily seen to be reasonable. Thus X is $c + 1$ -coreasonable according to Lemma 3.2.3(12).

Now we prove our claim for an irreducible X using Proposition 1.1.3. So, we need a small subcategory of $\mathcal{DK}(X)$ containing all “relevant objects”. We take \underline{C} being a small full triangulated subcategory of $\mathcal{DK}(X)$ containing all “geometrically defined” objects of this category; those are obtained from objects of the form $\mathbf{1}_D$ for D being separated of finite type over X by applying compositions of motivic image functors coming from finite type separated X -morphisms (cf. Remark 3.2.6 below). We fix a homological functor $F: \underline{C} \rightarrow R\text{-Mod}$ whose restriction to $VN_d(X)$ is zero; here we take $R = \mathbb{Q}$ if $\Lambda = \mathbb{Q}$ and $R = \mathbb{Z}_{(l)}$ for some $l \in \mathbb{P} \setminus \mathcal{S}$ in the opposite case. We should check that $F(\varphi_X^U(T \rightarrow X)) = \{0\}$.

Following §2 of [Kel14], in the case $\Lambda \neq \mathbb{Q}$ we apply [ILO14, Theorems IX.1.1 and Theorem II.4.3.2] to obtain a commutative diagram

$$\begin{array}{ccc} W' & \xrightarrow{k} & X' \\ \downarrow q & & \downarrow p \\ W & \xrightarrow{r} & X \end{array} \quad (3.3)$$

whose connecting morphisms are separated of finite type, p is proper and generically finite of degree prime to l , k is a Nisnevich cover, and W' is regular. Certainly, the schemes X' and W' are generically finite over X ; this

is also true for W (see Definition II.1.2.2 of *ibid.*). Thus X, W , and W' are of δ -dimension at most d according to Proposition 3.1.2. In the case $\Lambda = \mathbb{Q}$ we replace the usage of Theorem II.4.3.2 of [ILO14] by that of Theorem II.4.3.1 of *loc. cit.*; we obtain the same data with the only difference being that the generic degree of p may be arbitrary.

We fix some open dense regular $U \subset X$ such that $U' = U \times_X X'$ is regular and $\mathcal{O}_{U'}$ is a free \mathcal{O}_U -module. According to Lemma 3.2.3(12), it suffices to verify that the embedding $j: U \rightarrow X$ is $c + 1$ -coreasonable.

Now we apply the inductive assumption (as described above) to X and X' . We obtain the existence of closed $T \subset X$ and $T' \subset X'$ of δ -dimension at most $d - c$ such that $X \setminus T$ and $X' \setminus T'$ are reasonable. Combining this fact with Lemma 3.2.3(10, 3) we obtain the following: it suffices to verify the existence of an open reasonable neighborhood V of any generic point t of T in X . We will check this fact for some fixed t of this sort; for this purpose we will prove the existence of V such that $\varphi_V^{U \cap V}(Q \cap V \rightarrow V) \in VN_d(V)$.

Now we can “modify” (3.3) similarly to §6.2 of [CiD16]; note that the (easy) method of the proof of Lemma 4.2.14 of [CiD12] justifies this action without any difficulty. Since we can replace X by any open neighborhood V of t (replacing all other schemes in (3.3) by the corresponding pullbacks), we can assume that T' is finite over T and that the fiber of k over T' is an isomorphism (recall that k is a Nisnevich cover!). Now, $V \cap (X \setminus T)$ is reasonable according to part 2 of the lemma. Combining this fact with the triangle (3.2), part 12 of the lemma, and Remark 3.2.2(2) we reduce the statement in question to $\varphi_X^U(T \rightarrow U) \in VN_d(X)$.

Let $T_{W'}$ denote $T' \times_{X'} W'$, and denote the image $q(T_{W'})$ (noting that the restriction of q to T'_W is certainly finite) by T_W . We should verify the vanishing of $F(\varphi_X^U(T \rightarrow X))$. Now, Lemma 3.2.3(3) provides us with a chain of morphisms

$$\varphi(T \rightarrow X) \rightarrow \varphi(p^{-1}(T) \rightarrow X') \rightarrow \varphi(T' \rightarrow X') \rightarrow \varphi(T_{W'} \rightarrow W').$$

The results of applying F to these morphisms are injective according to the lemma. Being more precise, the induced homomorphism from $F(\varphi(T \rightarrow X))$ into $F(\varphi(p^{-1}(T) \rightarrow X'))$ is split injective according to part 9 of the lemma. Next, $F(\varphi(p^{-1}(T) \rightarrow X'))$ injects into $F(\varphi(T' \rightarrow X'))$ according to part 1 of the lemma; here we use the fact that $F(D) = 0$ since $D \in VN_d(X')$ according to the inductive assumption combined with Remark 3.2.2(1, 2). Lastly, $F(\varphi(T' \rightarrow X'))$ maps isomorphically onto $F(\varphi(T_{W'} \rightarrow W'))$ according to part 4 of the lemma.

It remains to note that the composition map $F(\varphi(T \rightarrow X)) \rightarrow F(\varphi(T_{W'} \rightarrow W'))$ factors through $F(\varphi(T_W \rightarrow W))$, whereas the latter group is zero according to part 11 of the lemma combined with the inductive assumption.

□

This statement has several nice implications.

Theorem 3.2.5. Let $f: Y \rightarrow X$ be a finite type (separated) morphism for X being a separated finite type B -scheme, and denote $\delta(Y)$ by d . Then the following statements are valid.

1. If Y is regular then there exists a morphism $\mathcal{MK}_X^{BM}(Y) \rightarrow f_*(\mathbf{1}_Y)$ whose cone belongs to $VN_r(X)$.
2. For any $r \geq 0$. $n \in \mathbb{Z}$, $-r - 1 \leq n \leq r$, $N_r(X)$ coincides with the envelope $N_r^n(X)$ of $\{\mathcal{MK}_X^{BM}(V)[m] : m \leq n\} \cup \{v_*(\mathbf{1}_V)[m] : m > n\}$ for $v: V \rightarrow X$ running through finite type (separated) morphisms with regular domain with $\delta(V) + |m| \leq r$.
3. $\mathcal{MK}_X^{BM}(Y)$ and $f_*(\mathbf{1}_Y)$ belong to $N_d(X)$.
4. We have $f_*(N_r(Y)) \subset N_r(X)$, whereas $f^!(N_r(X)) \subset N_r(Y)$ whenever f is quasi-finite.

Proof. 1. Recall that Y possesses an X -compactification \bar{Y} (see Remark 3.1.3(2)); denote the corresponding proper morphism $\bar{Y} \rightarrow X$ by \bar{f} . Then Lemma 3.2.3(1) gives a distinguished triangle $\mathcal{MK}_X^{BM}(Y) \rightarrow f_*(\mathbf{1}_Y) \rightarrow \bar{f}_!(\varphi_{\bar{Y}}^Y(Y))$; hence the statement follows from Proposition 3.2.4.

2. For $r = 0$ we have $\delta(V) = 0$ in the definitions of $N_r(X)$ and $N_r^n(X)$; thus V is proper over X . Hence the obvious inductive argument enables us to assume that $N_{r'}(X) = N_{r'}^n(X)$ for any $r' < r$ and $-r' - 1 \leq n' \leq r'$.

Next, it certainly suffices to verify that $N_r(X) = N_r^n(X)$ and that the classes $N_r^n(X)$ coincide for all n .

We certainly have $N_r(X) \subset N_r^n(X)$. The converse implication is immediate from Lemma 3.2.3(13).

Thus it remains to prove by descending induction on n that $N_r^{n-1}(X) = N_r^n(X)$ for any n between r and $-r$. This statement is also equivalent to $\mathcal{MK}_X^{BM}(V)[n] \in N_r^{n-1}(X)$ and $v_*(\mathbf{1}_V)[n] \in N_r^n(X)$ whenever V is a regular scheme of finite type over X and $|n| + \delta(V) \leq r$.

Now, assertion 1 yields that $v_*(\mathbf{1}_V)[n]$ belongs to the $\mathcal{DK}(X)$ -envelope of $\{\mathcal{MK}_X^{BM}(V)[n]\} \cup VN_{\delta(V)}[n]$. Hence to verify that $v_*(\mathbf{1}_V)[n] \in N_r^n(X)$ it suffices to check whether $VN_{\delta(V)}[n] \subset N_r^n(X)$. Next, the inductive hypothesis (with respect to n) yields that $N_r^n(X) = N_r(X)$, whereas the latter class contains $VN_{\delta(V)}[n]$ (if $\delta(V) + |n| \leq r$); here we apply Remark 3.2.2(1) for $n \geq 0$ and Lemma 3.2.3(13) for $n < 0$.

Now we verify that $\mathcal{MK}_X^{BM}(V)[n] \in N_r^{n-1}(X)$. The statement is obvious if $\delta(V) = 0$; see Remark 3.1.3(3).

Now we check our statement for $\delta(V) > 0$; so, $|n| < r$. Assume that $n > 0$; then $\mathcal{MK}_X^{BM}(V)[n-1] \in N^{r-1}(X)$. Applying the inductive hypothesis with respect to r we obtain that $\mathcal{MK}_X^{BM}(V)[n-1] \in N_{n-2}^{r-1}(X)$. Hence in this case it suffices to note that $N_{n-2}^{r-1}(X)[1] \subset N_{n-1}^r(X)$. Similarly, if $n < 0$ then $\mathcal{MK}_X^{BM}(V)[n+1] \in N^{r-1}(X)$; thus $\mathcal{MK}_X^{BM}(V)[n] \in N_n^{r-1}(X)[-1] \subset N_{n-1}^r(X)$.

It remains to consider the case $n = 0$. Applying assertion 1 once again, we note that $\mathcal{MK}_X^{BM}(V)$ belongs to the $\mathcal{DK}(X)$ -envelope of $\{v_*(\mathbf{1}_V)\} \cup VN^{\delta(V)}[-1]$. Thus it suffices to check that $VN^r[-1] \subset N_{-1}^r(X)$. Now, the class $VN^r[-1]$ equals the envelope of $N^{r-1}(X) \cup \{\mathcal{MK}_X^{BM}(V)[-1] : \delta(V) < r\}$ by definition. The latter envelope certainly lies in $N_{-1}^r(X)$.

3. The first part of the assertion is an immediate consequence of Remark 1.2.2(2). Applying it for $X = Y$ we obtain that $\mathbf{1}_Y \in N^d(Y)$. Hence $\mathbf{1}_Y \in N_{-d-1}^d(Y)$ (see assertion 2). It obviously follows that $f_*(\mathbf{1}_Y) \in N_{-d-1}^d(X)$, and it remains to apply assertion 2 once again.

4. Similarly to the argument above, we use the equalities $N^r(Y) = N_{-r-1}^r(Y)$ and $N^r(X) = N_{-r-1}^r(X)$. They imply the first part of the assertion immediately (cf. Remark 3.2.2(2)). Combining these equalities with Remark 1.2.2(6) we also obtain the second part of the assertion. \square

Remark 3.2.6. Note that the arguments of this section do not rely on Theorem 1.2.1(II.4). Thus Theorem 3.2.5(3) can be used as an important ingredient of its proof (cf. the proof of Corollary 6.2.14 of [CiD16]). This was our reason for considering (a small skeleton of) the category of “geometrically generated K -motives” instead of $\mathcal{DK}^c(X)$ in the proof of Proposition 3.2.4.

We describe some obvious consequences from the theorem.

Corollary 3.2.7. Let $f: Y \rightarrow X$ be a finite type morphism for X being a Λ -nice scheme; let d denote the dimension of an X -compactification of Y . Then $\mathcal{MK}_X^{BM}(Y)$ and $f_*(\mathbf{1}_Y)$ belong to the $\mathcal{DK}(X)$ -envelope of $\{\mathcal{MK}_X^{BM}(V)[m] : m \leq n, \dim(V) + |m| \leq d\} \cup \{v_*(\mathbf{1}_V)[m] : m > n, \dim(V) + |m| \leq d\}$, where V runs through all regular finite type X -schemes satisfying this inequality.

Proof. Once again, the symbol \overline{Y} will denote an X -compactification of Y ; it suffices to verify the assertion for $X = \overline{Y}$. Next, we take B equal to \overline{Y} also; then for the corresponding δ we certainly have $\delta(\overline{Y}) = \dim(\overline{Y}) = d$ (immediately from the definition of δ). Hence the assertion follows from Theorem 3.2.5(2, 3) combined with the inequality $\delta(-) \geq \dim(-)$ (see Proposition 3.1.2). \square

3.3 An application to bounding weights and to the vanishing of “too negative” \mathcal{K} -groups

Now we improve Corollary 2.3.2 (that includes Theorem 2.2.1 as a particular case) by combining Proposition 3.2.4 with Proposition 1.3.3(9).

Theorem 3.3.1. Let $f: Y \rightarrow X$ be a separated morphism of Λ -nice reasonable schemes, $\dim Y = d$, $0 \leq r \leq d$. Then the following statements are valid.

I.

1. $f_*(\mathbf{1}_Y) \in \mathcal{DK}(X)_{w_{\text{Chow}} \geq -d}$.
2. There exists a closed subscheme $Z \subset Y$ such that $\dim(Z) \leq r - 1$ and for the morphism $f': Y \setminus Z \rightarrow X$ we have $f'_*(\mathbf{1}_{Y \setminus Z}) \in \mathcal{DK}(X)_{w_{\text{Chow}} \geq r-d}$.
3. Assume that f is of finite type; let \bar{Y} be an X -compactification of Y (see Remark 3.1.3(2)) and denote its dimension by \bar{d} . Then $f_*(\mathbf{1}_Y) \in \mathcal{DK}(X)_{w_{\text{Chow}} \leq \bar{d}}$ and $\mathcal{MK}_X^{BM}(Y) \in \mathcal{DK}(X)_{w_{\text{Chow}} \geq -\bar{d}}$.

II.

1. For any smooth V/X and any open embedding $j: U \rightarrow V$ we have the following: $\mathcal{K}_s(V \times_X Y) = \{0\}$ for $s < -d$ and $\mathcal{K}_{-d}(V \times_X Y)$ surjects onto $\mathcal{K}_{-d}(U \times_X Y)$.
2. In the setting of assertion I.3 we have the following: for any (separated) finite type morphism $P \rightarrow X$ the groups $\mathcal{K}_{-s}^{BM,Y}(P \times_X Y)$ (see Remark 1.2.2(4)) vanish for all $s > r$ if (and only if) this statement is valid under the additional assumptions that P is regular and affine, $\dim(P) \leq \bar{d} - s$, and $s \leq d$.

Proof. I. All of the assertions easily follow the fact that for any Λ -reasonable scheme S and $v: V \rightarrow S$ being a morphism of finite type with a regular domain we have $\mathcal{MK}_S^{BM}(V) \in \mathcal{DK}(S)_{w_{\text{Chow}} \leq 0}$ and $v_*(\mathbf{1}_V) \in \mathcal{DK}(S)_{w_{\text{Chow}} \geq 0}$; see Proposition 2.3.1(2, 3, 4).

1. Indeed, applying Corollary 3.2.7 to the morphism id_Y , $n = -d - 1$ we deduce that $\mathbf{1}_Y = \mathcal{MK}_Y^{BM}(Y)$ belongs to $\mathcal{DK}(Y)_{w_{\text{Chow}} \geq -d}$. It remains to apply Proposition 2.3.1(3) once again.

2. We apply Theorem 3.2.5(2, 3) to the morphism id_Y obtaining that $\mathbf{1}_Y$ belongs to the envelope of $\{v_{m,*}(\mathbf{1}_{V_m})[m] : -d \leq m \leq d\}$ with $v_m: V_m \rightarrow Y$ being finite type morphisms with regular domains and $\delta^Y(V_m) \leq d - |m|$. We take Z being the scheme-theoretic image in Y of $\cup_{m > d-r} V_m$. Then $\dim(Z) \leq r - 1$ according to Remark 3.1.3(2). Next, for $U = Y \setminus Z$ and the open

embedding $u: U \rightarrow X$ we have $\mathbf{1}_U \cong u^*(\mathbf{1}_Y)$; thus $\mathbf{1}_U$ belongs to the $\mathcal{DK}(U)$ -envelope of $\{(v_m \times_Y u)_*(\mathbf{1}_{V_m \times_Y U})[m] : -d \leq m \leq d\}$, whereas $V_m \times_Y U$ is empty for $m > d - r$. Once again, it remains to apply Proposition 2.3.1(2, 4).

3. Immediate from Corollary 3.2.7 applied to f with n being equal to \bar{d} and $-\bar{d} - 1$, respectively.

II. 1. This is just a combination of assertion I.1 with Corollary 2.3.2.

2. The aforementioned corollary yields that the first condition in the assertion (i.e., the one without the restriction on $\dim(P)$) is fulfilled if and only if $f_*(\mathbf{1}_Y) \in \mathcal{DK}(X)_{w_{Chow} \geq -r}$.

Now we apply Proposition 1.3.3(9) for $M = f_*(\mathbf{1}_Y)$, $i = -r$. According to Corollary 3.2.7, M belongs to the envelope of $\{\mathcal{MK}_X^{BM}(V)[m] : m \leq -r - 1, \dim(V) - m \leq \bar{d}\} \cup \{v_*(\mathbf{1}_V)[m] : m \geq -r, \dim(V) + |m| \leq \bar{d}\}$, where V runs through all regular finite type X -schemes satisfying this inequality. Hence (combine the recollection in the beginning of the proof of assertion I with Proposition 1.3.3(9)) to verify that $f_*(\mathbf{1}_Y) \in \mathcal{DK}(X)_{w_{Chow} \geq -r}$ it suffices to check that $\mathcal{MK}_X^{BM}(V)[m] \perp M$ whenever V is regular scheme of finite type over X , $\dim(V) - m \leq \bar{d}$, and $m \leq -r - 1$. Now, this condition is automatic for $m > d$ according to assertion I.1. Thus it remains to recall that $\mathcal{DK}(X)(\mathcal{MK}_X^{BM}(V)[m], M) \cong \mathcal{K}_m^{BM,Y}(V \times_X Y)$ (see 1.6). □

Remark 3.3.2. 1. It appears that no analogues of part II.1 of the theorem were ever formulated in the literature; yet prof. S. Kelly has kindly informed the authors that the methods of [Kel14] may be used to prove a somewhat similar statement.

Yet part II.2 of the theorem is completely new. Note that one may certainly replace the Borel–Moore K -theory in it by relative K -theory and relate it to other conditions described in Theorem 2.2.1. Furthermore, one can slightly improve the statement by considering δ^X -dimensions instead of the Krull ones in it.

2. Part I.2 of the theorem also appears to be new. In particular, it says that for any $n \geq 0$ the property of $\mathbf{1}_X$ not to belong to $\mathcal{DK}(X)_{w_{Chow} \geq -n}$ is “supported in codimension $> n$ ”; thus the same is true for the non-vanishing of $\mathcal{K}_s(X)$ for $s < -n$ (along with the non-vanishing of $\mathcal{K}_{-s}^{BM,X}(P)$ for a separated finite type P/X). These facts are (more or less) trivial for $n = 0$; yet for $n > 0$ they do not seem to follow from Kelly’s results (since the inverse limit of $X \setminus Z$ for Z running through all closed subschemes of codimension greater than n does not exist as a scheme a general).

3. Now we explain that instead of checking that $\mathcal{K}_{-s}^{BM,Y}(P \times_X Y) = \{0\}$ for all (s, P) as in assertion II.2 it suffices to verify it for a certain set of “test pairs” (P_b, s_b) that may be explicitly described.

For this purpose it certainly suffices to verify the corresponding “explicit version” of Corollary 3.2.7, i.e., that $f_*(\mathbf{1}_Y)$ belongs to the envelope of $\{\mathcal{MK}_X^{BM}(V_b)[m_b] : m_b \leq n\} \cup \{v_*(\mathbf{1}_{v_p c})[m_c] : m > n\}$ for certain “explicit (V_b, m_b) ” (and V_b, V_c being regular of finite type over X , $\dim(V_b) + |m_b|$ and $\dim(V_c) + |m_c|$ being at most \bar{d}). Now we note that the proofs of the corresponding parts of Lemma 3.2.3 are quite “constructible”. Thus the proof of Proposition 3.2.4 can be used to construct explicit pairs of a similar sort in the cases $\Lambda = \mathbb{Z}_{(l)}$ (for some prime l) and $\Lambda = \mathbb{Q}$ since the arguments from [ILO14] used in it are constructible also. It easily follows that (V_b, m_b) can be explicitly described in this case.

To obtain an “explicit version” of Proposition 3.2.4 when Λ is distinct from \mathbb{Q} and $\mathbb{Z}_{(l)}$ one can start from applying the argument used in the proof of the proposition for $\Lambda' = \mathbb{Z}_{(l_0)}$ for some $l_0 \in p \setminus \mathcal{S}$. The (inductive) construction of the corresponding pairs for the object $\varphi_X^U(X)$ involves a finite number of applications of Lemma 3.2.3(9); denote the corresponding values of e by e_i . Then our arguments actually yield that the (“explicit”) envelope statement in question for $\varphi_X^U(X)$ is actually fulfilled in the category of motives with $\Lambda[(\prod e_i)^{-1}]$ -coefficients.

Next one can apply Corollary 0.2 of [BoS15a]. It states that for $M, M_j \in \text{Obj } \mathcal{DK}^c(X)$ the object M belongs to the envelope of $\{M_j\}$ whenever $c_{\mathcal{DK}^c(X)}^l(M)$ (see Remark 1.1.2(2)) belongs to the envelope of $\{c_{\mathcal{DK}^c(X)}^l(M_j)\}$ for any $l \in \mathbb{P} \setminus \mathcal{S}$.⁸ Thus for L being the set of prime divisors of $\prod e_i$ belonging to $\mathbb{P} \setminus \mathcal{S}$ the object $\varphi_X^U(X)$ belongs to the envelope of $\{\mathcal{MK}_X^{BM}(V_{b,X})^{l_0}[m_b^{l_0}]\} \cup (\cup_{l \in L} \{\mathcal{MK}_X^{BM}(V_{b,X}^l)[m_b^l]\})$ for the corresponding $V_{b,X}^l$.

This “explicit test statement” is completely new. Note also that in the next section we will relate the Chow-weights of K -motives with rational coefficients (and so, also the vanishing of the corresponding K -groups) to étale (co)homology of schemes.

4. The authors suspect that all the aforementioned properties of $\mathcal{K}(-)$ are valid for the “usual” (i.e., “non-homotopy-invariant”) K -theory. Unfortunately, the (“motivic”) methods of the current paper cannot be used to prove this conjecture (in the general case) since at the moment the

⁸ Recall that target of $c_{\mathcal{DK}^c(X)}^l$ is isomorphic to the $\mathbb{Z}_{(l)}$ -linear version of $\mathcal{DK}^c(X)$.

"theory of motivic categories" is understood well enough only for motivic categories satisfying the homotopy invariance property (see §3.4 below). Yet applying the "classical" comparison argument one immediately obtains the proof of the restriction of our conjecture (for $\mathbb{Z}[\frac{1}{p}]$ -linear K -theory) to nice schemes such that p is nilpotent on them; cf. (the proof of) [Kel14, Theorem 3.5].

5. The authors certainly do not claim that this method of constructing (V_b, m_b) is "optimal". One of the reasons for this is that one can "often" replace the usage of the aforementioned results of [ILO14] by much simpler (though less general) resolution of singularities results (in particular, the ones established in [Tem17]). So one can find a smaller set of (V_b, m_b) (at least) in some cases; cf. Proposition 4.5.1 below for an "easy" formulation of this sort (and Remark 4.3.6(3) for a very simple example).
6. One may also obtain certain "explicit test schemes" for bounding below the Chow-weights of $\text{Cone}(h_*(\mathbf{1}_{X''}) \rightarrow g_*(\mathbf{1}_{X'}))$ as mentioned in Remark 2.2.2(2); here one should combine the arguments above with Proposition 1.3.3(7). This corresponds to bounding from below the non-vanishing of certain relative Borel–Moore K -theory (and may be unfolded into the statements concerning some "bi-relative" homotopy invariant K -groups).
7. It would be interesting to combine (somehow) the arguments of the current paper with the recent work [KeS17].

3.4 Some "axiomatic" generalizations

We will now explain that our arguments can be applied to a wide range of "motivic" categories. Our main examples (other than K -motives) are Beilinson motives, cdh -motives (see Remark 1.2.2(8)), and h -motives as studied in [CiD16] (yet cf. Remark 3.4.3(1) below). We describe a certain axiomatic setup that would enable us to treat all of these settings simultaneously.

So, for a scheme B as above we will use the notation $G = G(B)$ to denote the category of separated finite type B -schemes (with morphisms being separated B -scheme morphisms). We assume the existence of a function δ from $\text{Obj } G$ into non-negative integers that fulfills all the properties listed in Proposition 3.1.2.

Next, we consider a 2-functor \mathcal{D} from G into the 2-category of (compactly generated) tensor triangulated categories that are closed with respect to small

coproducts. We assume that all the categories $\mathcal{D}(X)$ for $X \in \text{Obj } G$ are Λ -linear, where $\Lambda = \mathbb{Z}[\mathcal{S}^{-1}]$ and \mathcal{S} contains all the primes that are not invertible on B . All the notation for $\mathcal{D}(-)$ will be similar to that for K -motives (though we will sometimes put an index \mathcal{D} to avoid ambiguity). So, for $v: V \rightarrow X$ being a G -morphism we define ${}^{\mathcal{D}}\mathcal{M}_X^{BM}(V) = v_!({}^{\mathcal{D}}\mathbf{1}_V)$. We will assume that the \mathcal{D} -versions of Theorem 1.2.1(I.2–7) are fulfilled.

Moreover, we assume that for the splitting ${}^{\mathcal{D}}\mathcal{M}_X^{BM}(\mathbb{P}^1(X)) \cong {}^{\mathcal{D}}\mathbf{1}_X \oplus {}^{\mathcal{D}}\mathbf{1}_X\langle -1 \rangle$ coming from the natural morphisms $X \rightarrow \mathbb{P}^1(X) \rightarrow X$ the object ${}^{\mathcal{D}}\mathbf{1}_X\langle -1 \rangle$ is \otimes -invertible in $\mathcal{D}(X)$ for any $X \in \text{Obj } G$, and for any $i \in \mathbb{Z}$ the functor $-\langle i \rangle = - \otimes ({}^{\mathcal{D}}\mathbf{1}_-)^{\otimes i}$ “commutes” with all the functors of the type f^* , f_* , $f_!$, and $f^!$ for f being a G -morphism. Furthermore, in the setting of assertions I.(8–9) we assume that $f^! \cong f^*\langle s \rangle$ if f is everywhere of relative dimension s , and $i^!({}^{\mathcal{D}}\mathbf{1}_X)$ is isomorphic to ${}^{\mathcal{D}}\mathbf{1}_Z\langle -c \rangle$ if i is everywhere of codimension c .

Lastly, we assume that *homotopy invariance* holds for \mathcal{D} , i.e., that for any $X \in G$ and f being the projection $\mathbb{A}^1(X) \rightarrow X$ the counit morphism $f_!f^!({}^{\mathcal{D}}\mathbf{1}_X) \rightarrow {}^{\mathcal{D}}\mathbf{1}_X$ is invertible.

In order to describe the extension of the results of §3.2 to this general setting we need certain non-periodic generalizations of the corresponding definitions.

Definition 3.4.1. 1. In the notation of Definition 3.2.1 (including X being a B -scheme) we define ${}^{\mathcal{D}}\varphi_X^U(Z \rightarrow W) = \pi_*i_Z^*j_{W,*}({}^{\mathcal{D}}\mathbf{1}_{U_W})$.

In the case $W = X$, $Z = Q$ we will replace the notation ${}^{\mathcal{D}}\varphi_X^U(Q \rightarrow X)$ just by ${}^{\mathcal{D}}\varphi_X^U(X)$.

2. For $r \in \mathbb{Z}$ we define the following classes of objects in $\mathcal{DK}(X)$: $\widetilde{N}_r(X)$ is the $\mathcal{DK}(X)$ -envelope of $\{{}^{\mathcal{D}}\mathcal{M}_X^{BM}(V)[m]\langle -m \rangle\}$ for $V \rightarrow X$ running through G -morphisms with V being regular, $m \geq 0$, and $\delta(V) + m \leq r$; $\widetilde{VN}_r(X)$ is the envelope of $(\widetilde{N}_{r-1}(X)[1] \cup (\widetilde{N}_{r-1}(X))\langle -1 \rangle)$.
3. We will say that j is \mathcal{D} -reasonable if ${}^{\mathcal{D}}\varphi_X^U(X) \in \widetilde{VN}_{\delta(X)}(X)$.
4. For any $n \in \mathbb{Z}$, $-r - 1 \leq n \leq r$ we consider the envelope $\widetilde{N}_r^n(X)$ of $\{{}^{\mathcal{D}}\mathcal{M}_X^{BM}(V)\langle -\max(m, 0) \rangle[m] : m \leq n\} \cup \{v_*({}^{\mathcal{D}}\mathbf{1}_V)\langle -\max(m, 0) \rangle[m] : m > n\}$ for $v: V \rightarrow X$ running through finite type (separated) morphisms with regular domain and $\delta(V) + |m| \leq r$ (cf. Theorem 3.2.5(2)).

Now we will sketch the proof of the analogues of the results of §3.2; we will formulate them in the order in which they can actually be proven.

Theorem 3.4.2. Let $r \geq 0$; let X be a B -scheme.

- I. Then the following statements are valid.

1. The (obvious) \mathcal{D} -analogues of parts 1–8 of Lemma 3.2.3, Remark 3.2.2(2), and Lemma 3.2.3(10) are fulfilled.
2. $\widetilde{VN}_r(X) \subset \widetilde{N}_r(X)$. Moreover, for any $s > 0$ we have $\widetilde{VN}_{r-s}(X)\langle -s \rangle \subset \widetilde{VN}_r(X)$, $\widetilde{N}_{r-s}(X)\langle -s \rangle[-s] \subset \widetilde{VN}_{r-s+1}(X)\langle 1-s \rangle[-s] \subset \widetilde{N}_r(X)$, and $\widetilde{N}_{r-s}\langle -s \rangle[-1] \subset \widetilde{VN}_r$.
3. The \mathcal{D} -analogues of Lemma 3.2.3(11, 12) are also valid.

II Assume in addition that \mathcal{D} satisfies the following (*local*) *splitting property*: if $g: Y' \rightarrow Y$ is a degree $e > 0$ finite morphism of Λ -nice schemes then there exists an open $U \subset Y$ such that for the morphism $g_U = g \times_Y U: Y_U \rightarrow U$ the adjunction morphism ${}^{\mathcal{D}}\mathbf{1}_U \rightarrow g_{U,*}g_U^*({}^{\mathcal{D}}\mathbf{1}_U) = g_{U,*}({}^{\mathcal{D}}\mathbf{1}_{Y_U})$ becomes split injective (i.e., a coretraction) in the localized category $\mathcal{D}(U)[e^{-1}]$.

Then the \mathcal{D} -analogue of Theorem 3.2.5 is fulfilled.

Proof. I.1. Note that the \mathcal{D} -version of Theorem 1.2.1(I.7) implies that the functors f^* , f_* , $f^!$, and $f_!$ are invertible whenever f is a nil-immersion; see Proposition 2.3.6(1) of [CiD12]. Hence the proofs of the corresponding statements carry over to the \mathcal{D} -context without any difficulty.

2. Similarly to the proof of Lemma 3.2.3(13), the non-trivial inclusions can be easily obtained from the following observations: $\delta(\mathbb{G}_m(V)) = \delta((\mathbb{P}^1)(V)) = \delta(V)+1$, ${}^{\mathcal{D}}\mathcal{M}_X^{BM}(\mathbb{P}^1(V)) \cong {}^{\mathcal{D}}\mathcal{M}_X^{BM}(V)\langle -1 \rangle \oplus {}^{\mathcal{D}}\mathcal{M}_X^{BM}(V)$, and ${}^{\mathcal{D}}\mathcal{M}_X^{BM}(\mathbb{G}_m(V)) \cong {}^{\mathcal{D}}\mathcal{M}_X^{BM}(V)\langle -1 \rangle \oplus {}^{\mathcal{D}}\mathcal{M}_X^{BM}(V)[1]$ (the latter is an easy consequence of the homotopy invariance property).

3. The arguments used in the proofs of Lemma 3.2.3(11, 12) can be carried over to the \mathcal{D} -setting. The only notable distinction here is that when we consider regular stratifications of X and U , respectively, we should assume (for simplicity) that the components are irreducible for $l > 1$. Then for the corresponding embeddings $j_l: Y_l^\alpha \rightarrow Y$ (for Y equal to X or U , respectively) we would have $j_l^!({}^{\mathcal{D}}\mathbf{1}_Y) \cong {}^{\mathcal{D}}\mathbf{1}_{Y_l}\langle -\text{codim}_Y Y_l \rangle$. Still one can easily “handle” these Tate twists using the previous assertion.

II. It is easily seen that the local splitting property (along with assertion I) enables one to carry over all the arguments used in the proof of Proposition 3.2.4 and Theorem 3.2.5 to the \mathcal{D} -context. □

Remark 3.4.3. 1. In contrast with most of the results of this paper, Theorem 3.4.2 does not depend on the existence of any (Chow-type) weight structures. Thus it can be applied to h -motives as considered in [CiD16]. Since the latter are closely related to constructible complexes of étale sheaves, we obtain some new statements on the latter, that essentially

enhance Gabber’s constructibility results (see [ILO14, §XIII]). Note here that the methods of these papers allow us to avoid inverting the corresponding \mathcal{S} (in the coefficient ring Λ). One may also weaken some other of our restrictions on B .

2. Recall that in *ibid.* it was proved that the corresponding functors f^* , f_* , $f_!$, and $f^!$ respect the constructibility of objects. We could have used the method of *ibid.* (that were also applied in [CiD16]) to deduce the \mathcal{D} -version of this result from the \mathcal{D} -versions of Theorem 1.2.1(I.1, 11, 13) (combined with the assumptions on \mathcal{D} introduced above).

Note also that one can assume $\mathcal{D}(-)$ to be defined on all pro-open subschemes of $\text{Obj } G(B)$; then the \mathcal{D} -version of Theorem 1.2.1(I.14) can be used to reduce the local splitting property (that was described in Theorem 3.4.2(II)) to the following statement: if $g: F' \rightarrow F$ is a finite morphism of spectra of fields in $\text{pro-Obj } G(B)$ of degree e then the image of the unit morphism ${}^{\mathcal{D}}\mathbf{1}_F \rightarrow g_*g^*({}^{\mathcal{D}}\mathbf{1}_{F'}) \cong g_*({}^{\mathcal{D}}\mathbf{1}_{F'})$ in $\mathcal{D}(F)[e^{-1}]$ splits.

3. Now we illustrate our definitions in a rather simple (and yet quite important) case.

Assume that $\delta = \delta^B$ (see Definition 3.1.1(3)), whereas B is the spectrum of a perfect field. Then δ equals the Krull dimension function. Next, if we assume in addition that \mathcal{D} is the Beilinson motives 2-functor (as studied in [CiD12]) and X is finite over B , then $\mathcal{D}(X)$ is the unbounded version of the \mathbb{Q} -linear Voevodsky motives over X ; see §16.1 of [CiD12]. Moreover, for any smooth connected variety V/X the object $\mathcal{MK}_X^{\text{BM}}(V)\langle\delta(V)\rangle$ is the Voevodsky motif $\mathcal{M}_X(V)$ of V over X ; see [BoD17, (2.3.4.a)].

4. We also describe a simple case when $j_!({}^{\mathcal{D}}\mathbf{1}_U)$, $j_*({}^{\mathcal{D}}\mathbf{1}_U)$, and ${}^{\mathcal{D}}\varphi_X^U(X)$ can be put into “very explicit” envelopes as prescribed by Theorem 3.4.2. Assume that X is regular and that $Q = X \setminus U$ is the union $\cup_{j \in J} Q_j$, where all Q_j and all of the intersections $Q_{J'} = \cap_{j \in J'} Q_j$ for $J' \subset J$ are regular closed subschemes of X . Then one can easily verify the following: ${}^{\mathcal{D}}\mathcal{M}_X^{\text{BM}}(U)$ belongs to the envelope of $\{{}^{\mathcal{D}}\mathcal{M}_X^{\text{BM}}(Q_{J'})[-\#J'] : J' \subset J\}$, $j_*({}^{\mathcal{D}}\mathbf{1}_U)$ belongs to the envelope of $\{{}^{\mathcal{D}}\mathcal{M}_X^{\text{BM}}(Q_{J'})\langle -\text{codim}_X Q_{J'} \rangle[\#J'] : J' \subset J\}$, and the motif ${}^{\mathcal{D}}\varphi_X^U(X)$ belongs to the envelope of ${}^{\mathcal{D}}\mathcal{M}_X^{\text{BM}}(U) \cup \{{}^{\mathcal{D}}\mathcal{M}_X^{\text{BM}}(Q_{J'})\langle -\text{codim}_X Q_{J'} \rangle[\#J'] : J' \subset J, J' \neq \emptyset\}$. Note also that all the embeddings $Q_{J'} \rightarrow X$ are proper morphisms (and so, ${}^{\mathcal{D}}\mathcal{M}_X^{\text{BM}}(Q_{J'})\langle s \rangle$ is a “ \mathcal{D} -Chow motif” over X for any $s \in \mathbb{Z}$ and $J' \subset J$).

Hence these envelope statements demonstrate “explicitly” that ${}^{\mathcal{D}}\varphi_X^U(X) \in \widetilde{VN}_{\delta(X)}(X)$ and that the motives ${}^{\mathcal{D}}\mathcal{M}_X^{BM}(U)$ and $j_*({}^{\mathcal{D}}\mathbf{1}_U)$ belong to $\widetilde{N}_{\delta(X)}^n(X)$ for any n between $-\delta(X) - 1$ and $\delta(X)$ (so, we have no need to consider different envelopes for distinct n here). Certainly, this simple example motivated our definitions and formulations; it is quite remarkable that we are able to prove the corresponding results in our very general setting.

4 “Studying weights” using *ldh*-descent, Voevodsky motives, and their étale (co)homology

In this section we will mostly consider the case $\Lambda = \mathbb{Q}$ (and $\mathcal{S} = \mathbb{P}$); see Remark 4.1.2 below. So, it will be sufficient for us to assume all the schemes we consider to be nice (since this is equivalent to being \mathbb{Q} -nice).

In §4.1 we study the connecting functors between $\mathcal{DK}(-)$ (for $\Lambda = \mathbb{Q}$) with the ones of Beilinson motives. Since these functors are weight-exact (with respect to the corresponding Chow weight structures), the “weights” of $f_*(\mathbf{1}_Y)$ equal the ones of its Beilinson-motivic analogue $f_*(\mathbb{Q}_Y)$.

In §4.3 we relate the weights of $f_*(\mathbb{Q}_Y)$ to the étale cohomology of Y and the Deligne weights for it in the case where X is the spectrum of a field. The general formalism of weight spectral sequence yields an inequality between two weight bounds of this sort. This weight estimate is conjecturally precise; we prove that this is actually the case in certain “(almost) maximally singular” case (that we illustrate by a simple example).

In §4.4 we extend these results to the case of a (more) general X . Somewhat unfortunately, this forces us to consider certain perverse étale cohomology and weights for it. Note however that “the weights” of \mathbb{Q}_Y are not greater than that of $f_*(\mathbb{Q}_Y)$ and this inequality is precise in some cases; thus considering the case of a “general” X appears to be important for these matters.

In §4.5 we use an easy argument to put $f_*({}^{\mathcal{D}}\mathbf{1}_Y)$ in an envelope of an explicit set of shifts of Chow motives in the case where X is a variety (here one may take the motivic category $\mathcal{D}(-)$ being either to $\mathcal{DK}(-)$ or to any of the two aforementioned versions of Voevodsky motives). This certainly yields a “nice” explicit set of “test schemes” for the weights of $f_*({}^{\mathcal{D}}\mathbf{1}_Y)$. We also discuss a possible relation of our weight spectral sequences to the *singularity* ones of [PaP09] and [CiG14].

4.1 Comparison with the categories $DM(-)$

Now (in the case $\Lambda = \mathbb{Q}$) we compare $\mathcal{DK}(-)$ with the Beilinson motives $DM(-)$ (i.e., with \mathbb{Q} -linear Voevodsky motives).

Proposition 4.1.1. Let X be a nice scheme. Then the following statements are valid.

1. There exists an exact faithful functor $M_K(X): DM(X) \rightarrow \mathcal{DK}(X)$; it respects (small) coproducts and maps the unit object \mathbb{Q}_X of $DM(X)$ into $\mathbf{1}_X$.
2. The system of these functors (when X varies) is compatible with all the motivic image functors for the corresponding categories.
3. $M_K(X)$ possesses a faithful right adjoint $\mathcal{M}_{per}(X)$ that respects coproducts; these functors also commute with all the motivic image functors.
4. The composition $M_K(X) \circ \mathcal{M}_{per}(X)$ (resp. $\mathcal{M}_{per}(X) \circ M_K(X)$) sends $M \in \text{Obj } \mathcal{DK}(X)$ into $\coprod_{i \in \mathbb{Z}} M$ (resp. sends $N \in \text{Obj } DM(X)$ into $\coprod_{i \in \mathbb{Z}} N(i)[2i]$).

Proof. We apply several results from (§14.2 of) [CiD12].

First we note that instead of $SH(-)[\mathbb{P}^{-1}]$ one can consider the categories $D_{\mathbb{A}^1, \mathbb{Q}}(-)$ (see the formula (5.3.35.2) of *ibid.*); those are homotopy categories of certain “functorial” stable model categories. By Theorem 14.2.9 of *ibid.*, $DM(X)$ can be defined as the category of modules over a certain ring object $H_{B, X} \in \text{Obj } D_{\mathbb{A}^1, \mathbb{Q}}(X)$ (with respect to the underlying model structure). Next, $\mathcal{DK}(X)$ can be described as the category of modules over $\coprod_{i \in \mathbb{Z}} B_X(i)[2i]$ in the model category underlying $D_{\mathbb{A}^1, \mathbb{Q}}(X)$ (see Corollary 14.2.17 of *ibid.*). Hence we can set $M_K(X)(M) = \coprod_{i \in \mathbb{Z}} M(i)[2i]$ (for $M \in \text{Obj } \mathcal{DK}(X)$) and endow it with the natural structure of a $\coprod_{i \in \mathbb{Z}} B_X(i)[2i]$ -module; $\mathcal{M}_{per}(X)(N)$ can be described in terms of the corresponding forgetful functor. Certainly, both of these functors respect coproducts.

Our remaining assertions easily follow from the general formalism described in §7.2 of *ibid.* □

Remark 4.1.2. We have two reasons for concentrating on the case $\Lambda = \mathbb{Q}$ in this section.

1. The functors $\mathcal{M}_{per}(-)$ are defined in this setting only.
2. Below we will study the question when a bound from below on the “weights” of the étale homology of a motif M implies a similar bound on the

weights of M itself. Certainly, one may hope for a statement of this sort (for a more or less “general”) M only for motives with rational coefficients.

So, below we will apply several results from [Bon14] and [Bon15]. Note however that in [BoI15] similar statements for $\mathbb{Z}[\frac{1}{p}]$ -linear motives over characteristic p schemes were established (here we set $\mathbb{Z}[\frac{1}{p}] = \mathbb{Z}$ in the case $p = 0$).

As we have already said (see Remarks 2.3.3(2) and 2.1.3(2)), the categories $DM(-)$ are also endowed with Chow weight structures. Now we verify that $\mathcal{M}_{per}(-)$ and $M_K(-)$ “detect weights”.

Proposition 4.1.3. Let X be a nice scheme. Then for any $M \in \text{Obj } \mathcal{DK}(X)$ we have $M \in \mathcal{DK}(X)_{w_{Chow} \leq 0}$ (resp. $M \in \mathcal{DK}(X)_{w_{Chow} \geq 0}$) if and only if $\mathcal{M}_{per}(X)(M) \in DM(X)_{w_{Chow} \leq 0}$ (resp. $\mathcal{M}_{per}(X)(M) \in DM(X)_{w_{Chow} \geq 0}$).

Moreover, for any $N \in \text{Obj } DM(X)$ we have $N \in DM(X)_{w_{Chow} \leq 0}$ (resp. $N \in DM(X)_{w_{Chow} \geq 0}$) if and only if $M_K(X)(N) \in \mathcal{DK}(X)_{w_{Chow} \leq 0}$ (resp. $M_K(X)(N) \in \mathcal{DK}(X)_{w_{Chow} \geq 0}$).

Proof. First we recall that $M_K(X)$ “commutes with the motivic image functors” and sends \mathbb{Q}_X into $\mathbf{1}_X$. Hence the descriptions of the corresponding Chow weight structures yield the left weight-exactness of $M_K(X)$ and the right weight-exactness of $\mathcal{M}_{per}(X)$ (see Proposition 1.3.5).

Next, to verify the right weight-exactness of $M_K(X)$ we should check the following: if $N \in DM(X)_{w_{Chow} \geq 0}$ and $t: T \rightarrow X$ is a finite type morphism then $t_*(\mathbf{1}_T)[j] \perp M_K(X)(N)$ for all $j < 0$. The adjunction $M_K(X) \dashv \mathcal{M}_{per}(X)$ translates this into $t_*(\mathbb{Q}_T)[i] \perp \mathcal{M}_{per}(X) \circ M_K(X)(N)$. Hence it suffices to apply Proposition 4.1.1(4) (along with the compactness of \mathbb{Q}_T and the weight-exactness of $-(i)[2i]$ for $i \in \mathbb{Z}$).

Moreover, Proposition 1.3.5(6) yields that $\mathcal{M}_{per}(X)$ is left weight-exact. Hence both $\mathcal{M}_{per}(X)$ and $M_K(X)$ are weight-exact.

Next, Proposition 1.3.5(7) (along with part 4 of the previous proposition) yields that the “strict weight-exactness” of $M_K(X) \circ \mathcal{M}_{per}(X)$ and of $\mathcal{M}_{per}(X) \circ M_K(X)$ (this means: $M \in \mathcal{DK}(X)_{w_{Chow} \leq 0}$ if and only if $M_K(X)(\mathcal{M}_{per}(X)(M)) \in \mathcal{DK}(X)_{w_{Chow} \leq 0}$; $M \in \mathcal{DK}(X)_{w_{Chow} \geq 0}$ if and only if $M_K(X)(\mathcal{M}_{per}(X)(M)) \in \mathcal{DK}(X)_{w_{Chow} \geq 0}$; $M' \in DM(X)_{w_{Chow} \geq 0}$ if and only if $\mathcal{M}_{per}(M_K(X)(M')) \in \mathcal{DK}(X)_{w_{Chow} \geq 0}$; $M' \in DM(X)_{w_{Chow} \leq 0}$ if and only if $\mathcal{M}_{per}(M_K(X)(M')) \in \mathcal{DK}(X)_{w_{Chow} \leq 0}$). This certainly yields the results in question. □

Remark 4.1.4. In the current paper we are mostly interested in the “weights” of M being of the form $f_*(\mathbf{1}_Y)$ (for $f: Y \rightarrow X$ being a separated morphisms of nice schemes). According to the previous proposition, to compute the

“weight range” of (this) M it suffices to bound the weights of $\mathcal{M}_{per}(X)(M)$. Now, according to Proposition 4.1.1 the object $\mathcal{M}_{per}(X)(M)$ is isomorphic to $\bigoplus_{i \in \mathbb{Z}} f_*(\mathbb{Q}_Y)\langle i \rangle$. Thus Proposition 1.3.5(7) implies the following: for $n \geq 0$ we have $M \in \mathcal{DK}(X)_{w_{Chow} \geq -n}$ if and only if $f_*(\mathbb{Q}_Y) \in DM(X)_{w_{Chow} \geq -n}$.

4.2 On “detection of weights via étale (co)homology” over a field

As we have just shown (in the case $\Lambda = \mathbb{Q}$) it suffices to “study weights” for (compact) Voevodsky motives instead of K -ones; so we will only consider $DM(-)$ till the end of the paper.

Now we recall that (for any homological $H: DM_c(X) \rightarrow \underline{A}$) Corollary 1.4.3 yields a method for proving that a given (compact) motif N does not belong to $DM_c(X)_{w_{Chow} \geq -n}$ as follows:

Corollary 4.2.1. For some $M \in \text{Obj } DM_c(X)$ and $n \in \mathbb{Z}$ assume one of the following conditions is fulfilled:

1. $E_2^{pq} T_{w_{Chow}(X)}(H, M) \neq 0$ for some $q \in \mathbb{Z}$ and $p > n$.
2. $(W_{-n-1} H_q)(M) \neq 0$ for some $q \in \mathbb{Z}$.

Then $M \notin DM_c(X)_{w_{Chow}(X) \geq -n}$. Moreover, condition 2 implies condition 1.

So the problem is to find such an H for which the corresponding computations are manageable.

In §4.4 below we will consider two different H related to étale homology.⁹ We will start with the case $X = \text{Spec } k$, k is a field; in this case our versions of H coincide.

We fix a prime l ; till the end of the section we will assume that it is invertible on all the schemes we consider (so, $l \neq \text{char } k$ if $X = \text{Spec } k$). Now recall the existence of a functor $\mathcal{H}_{\mathbb{Q}_l}^{et}(X): DM_c(X) \rightarrow D_c^b Sh^{et}(X, \mathbb{Q}_l)$; the latter is the triangulated category of constructible étale complexes of \mathbb{Q}_l -sheaves (i.e., of continuous finite dimensional \mathbb{Q}_l -representations of the absolute Galois group of k). Since we will apply a generalization of this functor below, the optimal reference for us here is §7.2 of [CiD16]. Let $H_{\mathbb{Q}_l}^{et} = H_{\mathbb{Q}_l}^{et}(X)$ denote the composition of $\mathcal{H}_{\mathbb{Q}_l}^{et}(X)$ with the zeroth canonical truncation functor for $D_c^b Sh^{et}(X, \mathbb{Q}_l)$; so, the target of $H_{\mathbb{Q}_l}^{et}$ is the category the corresponding $Sh^{et}(X, \mathbb{Q}_l)$.

Now we describe a certain “weight detection” conjecture; we will generalize it later.

⁹We restrict our theories to compact motives since étale homology probably does not behave “nicely enough” on the whole $DM(X)$; cf. Lemma 2.4 of [Ayo15].

Conjecture 4.2.2. Let $r, n \geq 0$. We will say that the conjecture $WD^n(X)$ holds if for $M \in \text{Obj } DM_c(X)$ we have $M \in DM_c(X)_{w_{Chow} \geq -n}$ whenever $E_2^{pq}T_{w_{Chow}(X)}(H_{\mathbb{Q}_l}^{et}, M) = 0$ for all $q \in \mathbb{Z}$ and $p > n$.

We will say that $WD_r^n(X)$ is valid if this implication holds under the additional assumption that $M \in \tilde{N}_r(X)$ (see Definition 3.4.1(2); we take $\mathcal{D} = DM(-)$ and δ being the Krull dimension function in it).

Proposition 4.2.3. The following statements are valid for any $n \geq 0$.

1. $V_r(X)$ equals the envelope of $\{\mathcal{MK}_X^{BM}(V)[m]\langle -\max(m, 0) \rangle\}$ for V running through proper regular k -schemes and $\dim(V) + |m| \leq r$. In particular, if k is perfect then $V_r(X)\langle r \rangle$ equals the envelope of $\{\mathcal{M}_X(V)[m]\langle -\min(m, 0) \rangle\}$ for V being smooth projective over k and $\dim(V) + |m| \leq r$.
2. Conjecture $WD^n(X)$ is equivalent to the combination of the conjectures $WD_r^n(X)$ for all $r \geq 0$.
3. Let k' be an algebraic extension of k , $X' = \text{Spec } k'$. Then for any $r \geq 0$ the conjecture $WD_r^n(X')$ implies $WD_r^n(X)$.
4. Conjecture $WD_r^n(X)$ holds if $n \geq r - 2$.

Proof. **1.** The ‘‘in particular’’ part of the assertion is an easy implication of the first part; see Remark 3.4.3(3). Now, the first part of the assertion can be easily verified using the alteration arguments from §2.4 of [BoD17].

2. Easy; note that $\text{Obj } DM_c(X)$ equals the unions of $\tilde{N}_r(X)\langle j \rangle$ and of $\tilde{N}'_r(X)\langle j \rangle$ for all $j \in \mathbb{Z}$ and $r \geq 0$.

3. Denote by f the corresponding morphism $X' \rightarrow X$. According to Remark 7.2.25 of [CiD16], there exists a commutative square of functors

$$\begin{array}{ccc} DM_c(X) & \xrightarrow{H_{\mathbb{Q}_l}^{et}(X)} & Sh^{et}(X, \mathbb{Q}_l) \\ \downarrow f^* & & \downarrow f_{et}^* \\ DM_c(X') & \xrightarrow{H_{\mathbb{Q}_l}^{et}(X')} & Sh^{et}(X', \mathbb{Q}_l) \end{array} \quad (4.1)$$

with f_{et}^* being an exact conservative functor. Thus it suffices to verify that $M \in DM_c(X)_{w_{Chow} \geq -n}$ whenever $f^*(M) \in DM_c(X')_{w_{Chow} \geq -n}$. The latter statement is a particular case of Theorem 2.3.1(VI) of [Bon15].

4. We should verify that if $M \in V_r(X)$ and $E_2^{pq}T_{w_{Chow}(X)}(H_{\mathbb{Q}_l}^{et}, M) = 0$ for all $q \in \mathbb{Z}$ and $p > n$, then $M \in DM_c(X)_{w_{Chow} \geq -n}$ (assuming that $n \geq r - 2$).

Certainly, the statement is trivial for $n \geq r$; so we consider the cases $n = r - 1$ and $n = r - 2$.

According to the previous assertion, we can (and will) assume that k is algebraically closed. Now, we consider $t(M) = (M^i)$. According to Proposition 1.4.1(4) we may assume that $M^i = 0$ for $i > r$, and for any $i \geq 0$ the motif M^{r-i} is a retract of $\mathcal{M}_X(V^{r-i})\langle -i \rangle$ with V^{r-i} being smooth projective of dimension at most i over k (see Remark 4.2.3(2) of [BoS15b]). According to 1.4.1(3) we should check that $t(M)$ is homotopy equivalent to a complex concentrated in degrees $\leq n$.

Assume now that $n = r - 1$. We should verify that the boundary morphism $d^{r-1}: M^{r-1} \rightarrow M^r$ splits, whereas our condition on the weight spectral sequence translates into the assumption that d^{r-1} induces surjections on the \mathbb{Q}_l -étale homology in all degrees. This is a very simple property of Chow motives that is easily seen to follow from the fact that the numerical equivalence coincides with the homological one for 0-cycles on $V^r \times V^{r-1}$ (since V^r is just a collection of k -points).

Now assume that $n = r - 2$. According to (just established) case $n = r - 1$ of our assertion, we may assume that $M^r = 0$. We should check that d^{r-2} splits. We certainly can assume that all the components of V^{r-i} are of dimension i (for $i = 1, 2$). Consider (for convenience) the dual d to d^{r-2} ; so this is a morphism from a retract M_1 of $\mathcal{M}_X(V^{r-1})$ into a retract M_2 of $\mathcal{M}_X(V^{r-2})$ that yields surjections on the étale cohomology H_{et, \mathbb{Q}_l}^* in all degrees. We should verify that d is split injective.

For $s = 0, 1$ consider the full additive subcategories C_s of $Chow^{eff}$ consisting of $N \in \text{Obj } Chow^{eff}(k)$ such that $H_{\mathbb{Q}_l}^{et, j}(N) = 0$ for $j \neq s$; we define C_2 as the subcategory containing those $N \in \text{Obj } Chow^{eff}$ such that $H_{et, \mathbb{Q}_l}^j(M) = 0$ for $j < 2$.

Certainly, C_0 is equivalent to the category of finite dimensional \mathbb{Q} -vector spaces and C_1 is equivalent to the Karoubi envelope of the \mathbb{Q} -linearization of the category of k -abelian varieties; thus both of them are abelian semi-simple. We also have $C_s \perp C_t$ for $0 \leq t < s \leq 2$ (see [B-VK16, Theorems 6.2.1, 9.2.2, and 15.3.1, and Proposition 17.5.4]). Moreover, any object of $Chow^{eff}$ can be presented as the direct sum of objects of C_s (for $s = 0, 1, 2$).

It follows that d can be presented as a successive extension of certain C_s -morphisms d_s in for $s = 2, 1, 0$. Hence the cohomological characterization of C_i allows to consider the case $d = d_s$ for a single s .

If $s = 0$ or 1 then our assertion immediately follows from the semi-simplicity of C_s along with the well-known fact that H_{et, \mathbb{Q}_l}^s is conservative on C_s .

It remains to consider the case $s = 2$. Since V_{r-1} is of dimension at

most 1, M_1 is a direct sum of $\mathbb{Q}_k\{1\}$ (recall that we assume k to be algebraically closed). Hence the fact that numerical equivalence coincides with the homological one for divisors easily implies that d splits. \square

Remark 4.2.4. 1. According to Remark 2.4.3 of [Bon10a] (note that it is no problem to apply it to homology instead of cohomology), the spectral sequence $E_2^{pq}T_{w_{\text{chow}}(X)}(H_{\mathbb{Q}_l}^{et}, M)$ degenerates at E_2 , and its E_2 -terms are the corresponding factors of a certain version of Deligne’s weight filtration on $H_{\mathbb{Q}_l}^{et}(M)$. We will make this much more precise below.

2. Certainly, the argument used in the proof of part 4 of the proposition also allows to reduce Conjecture 4.2.2 to the Beilinson’s conjectures on mixed motives and their weights (see [Bei87]). Many more statements of this sort can be found in [Bon15].

Note also that in the case $k \subset \mathbb{C}$ (and so, $\text{char } k = 0$) one may consider singular (co)homology instead of étale one (here one may use Saito’s Hodge modules and the “realization” results of [Ivo16]). In this case one may also relate our conjecture to the Hodge one; see Proposition 7.4.2 of [Bon09]. Moreover, one can probably relate the corresponding spectral sequence to the *singularity* one constructed in §5 of [CiG14]; see Remark 4.5.2(3) below.

4.3 On the relation to Deligne’s weights and “maximally singular” examples

Certainly, the case most interesting for the purposes of the current paper is $M = f_*(\mathbb{Q}_Y)$ for Y being a separated scheme of finite type over X . Note that according to Theorem 3.4.2(II), if $d = \dim(Y)$ then $M \in V_d(X)$.

Since Deligne’s weights on étale cohomology are defined only if k is a finitely generated field (where $X = \text{Spec } k$), we will start from computing $E_2^{pq}T_{w_{\text{chow}}(X)}(H_{\mathbb{Q}_l}^{et}, M)$ in this case.

Proposition 4.3.1. Assume that $M = f_*(\mathbb{Q}_Y)$ for a morphism $f: Y \rightarrow X$ of reasonable schemes. Then the following statements are valid.

1. $\mathcal{H}_{\mathbb{Q}_l}^{et}(M)$ is canonically isomorphic to the étale total direct image $Rf_{*,et}(\mathbb{Q}_{lY})$.

2. Assume that $X = \text{Spec } k$ where k is a finitely generated field (so, it is finitely generated over the corresponding prime field), f is a separated morphism of finite type. Then $E_2^{pq}T_{w_{\text{chow}}(X)}(H_{\mathbb{Q}_l}^{et}, M)$ is canonically isomorphic to the q th weight factor of the étale cohomology sheaf $R^{p+q}f_{*,et}(\mathbb{Q}_{lY})$.

Proof. 1. Once again, we apply the (2)-functoriality of the family of functors $\mathcal{H}_{\mathbb{Q}_l}^{et}(Y)$ (as given by §7.2 of [CiD16] that extends the earlier results of [Ayo14], cf. also Theorem 2.4.1(I) of [Bon15]). The assertion is immediate from the compatibility of $\mathcal{H}_{\mathbb{Q}_l}^{et}(-)$ with functors of the type f_* (cf. (4.1)).

2. According to the previous assertion, the spectral sequence $T_{w_{Chow}(X)}(H_{\mathbb{Q}_l}^{et}, M)$ converges to $R^{p+q}f_{*,et}(\mathbb{Q}_{lY})$. We should prove that it degenerates and its E_2 -terms are the corresponding weight factors of $R^{p+q}f_{*,et}(\mathbb{Q}_{lY})$.

By the definition of Deligne’s weights, we should choose some “models” $\tilde{f}: \tilde{Y} \rightarrow \tilde{X}$ (i.e., X should be the generic point of \tilde{X} , \tilde{f} is a separated morphism of finite type of separated schemes) such that \tilde{X} is of finite type over $\text{Spec } \mathbb{Z}[\frac{1}{l}]$, and the sheaves $R^j \tilde{f}_{*,et}(\mathbb{Q}_{l\tilde{Y}}) \cong H_{\mathbb{Q}_l}^{et,j}(M)$ are punctually mixed for all $j \in \mathbb{Z}$ (i.e., for any closed embedding of a point $i_x: x \rightarrow \tilde{X}$ the sheaves $i_{x,*}^{et}(R^j \tilde{f}_{*,et}(\mathbb{Q}_{l\tilde{Y}}))$ are endowed with compatible filtrations with pure factors).

We start from choosing arbitrary models $f': Y' \rightarrow X'$ (with X' being reduced of finite type over $\text{Spec } \mathbb{Z}[\frac{1}{l}]$); for $M' = f'(\mathbb{Q}_{Y'})$ consider some choice of $t_{w_{Chow}(X')}(M') = (M'^i)$. According to Remark 2.3.7(4) of [Bon14], there exists an open (dense) embedding $j': X'' \subset X'$ such that $j'^*(M'^i)$ are certain retracts of $p_*^i(\mathbb{Q}_{P^i})$, where p^i are some compositions of universal homeomorphisms with smooth proper morphisms and P^i are regular. Moreover, the continuity property for $DM_c(-)$ (cf. Theorem 1.2.1(I.14)) yields the existence of an open dense $\tilde{X} \subset X''$ such that for the $\tilde{j}: \tilde{X} \rightarrow X'$ the corresponding retracts of $(p^i \times_{X''} \tilde{X})_*(\mathbb{Q}_{P^i \times_{X''} \tilde{X}})$ are the terms of (some choice of a) $w_{Chow(\tilde{X})}$ -weight complex of $\tilde{j}^*(M')$ (see Proposition 1.4.1(5)). Now, the sheaves $R^j(p^i \times_{X''} \tilde{X})_{*,et}(\mathbb{Q}_{P^i \times_{X''} \tilde{X}})$ are certainly punctually pure of weight j for any i and all $j \in \mathbb{Z}$. Since the categories of punctually pure sheaves of a given weight contain all subquotients of their objects, and there are “no non-zero morphisms between distinct weights”, Remark 2.4.3 of [Bon10a] yields the result (note once again that it is no problem to apply it to homology instead of cohomology). □

Remark 4.3.2. Certainly, the argument above may be applied to any $M \in \text{Obj } DM_c(X)$.

Moreover, it is actually not (that) necessary to assume that k is a finitely generated field. Indeed, the continuity property of $DM_c(-)$ ensures the existence of a finitely generated $k_0 \subset k$ such that M “is defined over k_0 ”, i.e., for the corresponding morphism $g: X \rightarrow X_0 = \text{Spec } k_0$ there exists $M_0 \in \text{Obj } DM_c(X_0)$ such that $g^*(M_0) \cong M$. Now, the functor g^* is weight-exact (with respect to the corresponding Chow weight structures) according to Theorem 2.3.1(VI) of [Bon15]. Thus the factors of the Chow-weight filtra-

tion for $H_{\mathbb{Q}_l^*}^{et}(M)$ “may be computed at X_0 ”; see the proof of [Bon15, Theorem 2.5.4(II.1)] for more detail.

Next, Proposition 4.1.3 justifies the usage of the notation $c(Y/X)$ (see Remark 2.2.2(1)) for Beilinson motives instead of K -ones.

Proposition 4.3.3. For $f: Y \rightarrow X$ being any separated morphism of nice schemes we set $c(Y/X) = n$ whenever

$$g_*(\mathbb{Q}_Y) \in DM(X)_{w_{Chow} \geq -n} \setminus DM(X)_{w_{Chow} \geq 1-n}.$$

I. Then the following statements are valid.

1. $c(Y/X) \geq 0$.
2. $c(Y/X) \leq c(Y/Y)$.
3. $c(Y/Y) \leq \dim(Y)$.
4. More generally, for any $r \geq 0$ there exists $Z \subset Y$ of dimension less than r such that $c((Y \setminus Z)/X) \leq \dim(Y) - r$.
5. Let $x_i: X_i \rightarrow X$ be an open cover of X . Then $c(Y/X)$ equals the maximum of $c(Y_i/X_i)$ for $Y_i = Y \times_X X_i$.

II. Assume that $X = \text{Spec } k$. Then $c(Y/X) \leq n$ for $n \geq \dim(Y) - 2$ if and only if $E_2^{pq} T_{w_{Chow}(X)}(H_{\mathbb{Q}_l^*}^{et}, f_*(\mathbb{Q}_Y)) = 0$ for all $p > n$.

Proof. I. The $\mathcal{DK}(X)$ -analogues of assertions 1–4 are given by Remark 2.2.2(1) and Theorem 3.3.1(I.1–2), respectively, whereas to obtain the $\mathcal{DK}(X)$ -version of assertion 5 one should apply Proposition 2.3.1(9) to the motif $f_*(\mathbf{1}_Y)$.

To carry over these statements to Beilinson motives it remains to apply Propositions 4.1.1 and 4.1.3.

Besides, assertions 1, 2, and 5 can be easily deduced from the results of [Bon14], whereas assertions 3 and 4 follow from Theorem 3.4.2(II).

II This is just a part of Proposition 4.2.3(4). □

Remark 4.3.4. 1. Moreover, we certainly have $c(Y/Y) = c(Y/X) = 0$ whenever Y_{red} is regular. Thus one may say that $c(Y/Y)$ is a certain “measure of singularity” for Y . We will say that Y is *maximally singular* whenever $c(Y/Y) = \dim(Y)$. Note that (according to parts I.4–5 of our proposition) this property is “concentrated around” a finite number of closed points of Y .

2. Certainly, $c(Y/X) = \dim(Y)$ is an even stronger restriction on Y (say, for X being the spectrum of a field). Yet we will now demonstrate that this equality is not a “local” condition.

3. It certainly may be interesting to relate our results (along with the maximal singularity condition for schemes) to Theorem 7.1 of [Has04]. As an intermediate step, one may try to consider a cohomology theory on $DM(X)$ or on $\mathcal{DK}(X)$ defined similarly to the *weight homology* of [KeS16].

We will study in detail two simple examples for our notions. We set $X = \text{Spec } k$ for some infinite field k (actually, this infiniteness restriction can easily be dropped), fix some $r \geq 0$, $N \geq 1$, and denote by Y_N the union of some collection of N generic hyperplanes in $\mathbb{A}^{r+1}(k)$ (we want any subset of our collection of hyperplanes to intersect properly; in particular, this means that any two non-empty intersections of distinct subsets of our collection are distinct).

Proposition 4.3.5. 1. $c(Y_N/X) = r$ whenever $N > 2r$.

2. $c(Y_{r+1}/X) \leq \max(r - 2, 0)$.

3. Y_N is maximally singular for any $N > r$.

Proof. 1,2. Combining Proposition 4.2.3(4) with Proposition 4.3.1 (along with Remark 4.3.2), it suffices to compute the quotients of the weight filtration for $H_{et, \mathbb{Q}_l}^*(Y_N)$. This is easy; note that these weights can be computed using the spectral sequence $E_1^{pq} = \bigoplus_{I \subset [1..N], \#I=p+1} H_{et, \mathbb{Q}_l}^q(Y_I) \implies H_{et, \mathbb{Q}_l}^{p+q}(Y_N)$. coming from proper descent, where Y_I is the intersection of the hyperplanes whose numbers belong to I . Certainly, the only cohomology of Y_I is \mathbb{Q}_l in degree 0; so all the Deligne’s weights in $H_{et, \mathbb{Q}_l}^*(Y_N)$ are zero. Thus it suffices to note that $H_{et, \mathbb{Q}_l}^r(Y_N) \neq 0$ whenever $N > 2r + 1$ and that $H_{et, \mathbb{Q}_l}^j(Y_{r+1}) = 0$ for all $j \neq 0$.

3. According to assertion 1, Y_{2r+1} is maximally singular. Now, for any $N_1, N_2 > r$ the schemes Y_{N_i} admit open covers such that any component of the first cover is isomorphic to some component of the second one and vice versa. Hence the statement follows from Proposition 4.3.3(5). □

Remark 4.3.6. 1. Thus the criterion given by Proposition 4.2.3(4) (along with Proposition 4.3.1) is non-vacuous, and our results yield quite a non-trivial relation between (weights of) étale (co)homology and negative \mathcal{K} -groups! Note moreover that the corresponding cycle class maps (i.e., “regulators”) are very far from being surjective or injective (in general).

2. On the other hand, we obtain that one cannot compute $c(Y/Y)$ looking at the X -cohomology of Y only. This urges us to consider the relative versions of our criteria.

3. Now denote Y_{r+1} by Y . It can be easily seen (cf. the proof of our proposition and of Proposition 4.5.1 below) that there exists a choice of a weight complex (M^i) of $M = \mathbb{Q}_Y$ such that $M^i = \bigoplus_{I \subset [1..N], \#I=-i+1} \mathcal{M}_{DMY}^{\text{BM}}(Y_I)$; in particular, $Y_{-r} = \mathcal{M}_{DMY}^{\text{BM}}(f)$ for f being the intersection of all the hyperplanes. Combining the latter fact with Proposition 1.3.3(9) and with part 3 of our proposition we obtain that $\mathcal{K}_{-r}^{\text{BM},Y}(f) \neq \{0\}$ (cf. Remark 3.3.2(3)); so, f is the corresponding "test scheme". We can re-formulate this condition using the long exact sequence

$$\mathcal{K}_{-r+1}(Y) \rightarrow \mathcal{K}_{-r+1}(Y \setminus f) \cdots \rightarrow \mathcal{K}_{-r}^{\text{BM},Y}(f) \rightarrow \mathcal{K}_{-r}(Y) \rightarrow \dots$$

(see (1.6)); we obtain that Y being maximally singular is "seen from" the non-surjectivity of $\mathcal{K}_{-r+1}(Y) \rightarrow \mathcal{K}_{-r+1}(Y \setminus f)$. Moreover, for $r > 1$ the latter fact is equivalent to the non-vanishing of $\mathcal{K}_{-r+1}(Y \setminus f)$ according to part 2 of our proposition.

4. Actually, one may apply the motivic Verdier duality (see §4.4 of [CiD12]) to prove that $c(Y_{r+1}/X) = 0$.

4.4 “Detection of weights” via étale homology over a general base

Now we discuss two “relative” generalizations of Conjecture 4.2.2. For this purpose we fix some $\mathbb{Z}_{(l)}$ -nice base scheme B , and denote by G' the set of schemes that are separated and essentially of finite type over B .¹⁰ We will need certain (non-negative) dimension functions on G' ; for our two homology theories two somewhat different types of dimension functions are “optimal”. So, let δ denote the function obtained by the trivial extension of Definition 3.1.1 from $G(B)$ to $G'(B)$.

Now, our second homology theory is a certain perverse étale homology for motives. The latter was defined in §2.4 of [Bon15] (certainly, this section heavily relied on [CiD16]); yet we note that in the setting of “general” schemes one should invoke a “true” dimension function (see Remark 3.1.3(1)) in this construction. So we assume that a non-negative function δ' is defined on G' , and this function satisfies the natural analogues of the properties listed in Proposition 3.1.2 along with $\delta(X) - \delta'(Z) = \text{codim}_X(Z)$ for $Z \subset X$ being any irreducible schemes in G' (cf. Remark 3.1.3(1)). Recall that the existence

¹⁰We consider pro-open subschemes of (finite type) B -schemes since we want to pass to certain limits in the theorem below.

of δ' is a certain restriction on B (we will abbreviate it by saying that “ δ' exists”); however, it suffices to assume that B is irreducible.

For any $Y \in G'$ we consider the \mathbb{Q}_l -étale realization functor $\mathcal{H}_{\mathbb{Q}_l}^{et}(Y): DM_c(Y) \rightarrow D^cSh^{et}(Y, \mathbb{Q}_l)$ (this is an exact functor whose target is a certain version of the derived category of constructible \mathbb{Q}_l -étale sheaves over X , as defined in §7.2 of [CiD16] (generalizing “classical” l -adic categories).¹¹ Next, $D^cSh^{et}(Y, \mathbb{Q}_l)$ is endowed with

a bounded perverse t -structure (for the “self-dual” perversity corresponding to δ') that will be denoted just by t ; the heart of t will be denoted by $Sh_{per}^{et}(Y, \mathbb{Q}_l)$. We set $H_{\mathbb{Q}_l}^{et}(Y)$ to be the composition of $\mathcal{H}_{\mathbb{Q}_l}^{et}(X)$ with the zeroth t -homology functor on $D^cSh^{et}(Y, \mathbb{Q}_l)$.

A special case here is when Y is the spectrum of a field. Since in this case the perverse t -structure differs from the canonical one only by a shift, one may (at the price of a minor abuse of notation) take for $H_{\mathbb{Q}_l}^{et}(Y)$ the functor introduced in the previous subsection. In particular, this functor can be defined without (necessarily) assuming that δ' exists.

Actually, we will mostly be interested in δ' in the case where B is the spectrum of a field (and then it coincides with δ , and its restriction to $G(B)$ is just the Krull dimension function) and for X being of finite type over B ; cf. Theorem 4.4.2(4, 6) below.

Now we are able to formulate our “relative weight detection” conjectures.

Conjecture 4.4.1. Let $X \in G'$, $r, n \geq 0$.

1. We will say that the conjecture $WD^n(X)$ holds if for any for $M \in \text{Obj } DM_c(X)$ and H being the product of the theories $H_{\mathbb{Q}_l}^{et}(x) \circ i_x^!$, where x runs through Zariski points of X , i_x is the corresponding embedding (see Proposition 2.3.1(11) for the definition of $i_x^!$), we have $M \in DM_c(X)_{w_{Chow} \geq -n}$ whenever $E_2^{pq}T_{w_{Chow}(X)}(H, M) = 0$ for all $q \in \mathbb{Z}$ and $p > n$.

We will say that $WD_r^n(X)$ is valid if this implication holds under the additional assumption that $M \in \tilde{N}_r(X)$ (see Definition 3.4.1(2); we take $\mathcal{D} = DM$ in it).

2. Assume that δ' is defined (in the sense described above). Then we will say that the conjecture $WD'^n(X)$ holds if for $M \in \text{Obj } DM_c(X)$ we have $M \in DM_c(X)_{w_{Chow} \geq -n}$ whenever $E_2^{pq}T_{w_{Chow}(X)}(H_{\mathbb{Q}_l}^{et}(X), M) = 0$ for all $q \in \mathbb{Z}$ and $p > n$.

¹¹This realization construction extends a closely related one from [Ayo14].

We will say that $WD_r^n(X)$ is valid if the latter implication holds under the additional assumption that $M \in \widetilde{N}'_r(X)$; here $\widetilde{N}'_r(X)$ is defined by setting $\mathcal{D} = DM$ and $\delta = \delta'$ in Definition 3.4.1(2).

Now we describe some evidence supporting these conjectures; we will also discuss some conjectural evidence for them later.

Theorem 4.4.2. The following statements are valid.

1. Conjecture $WD^n(X)$ is equivalent to the combination of the conjectures $WD_r^n(X)$ for all $r \geq 0$.
2. $WD^m(X)$ is equivalent to the combination of all $WD_r^m(X)$.
3. Conjecture $WD_r^n(X)$ holds whenever $WD_r^n(x)$ does for x running through all Zariski points of X .
4. Conjecture $WD_r^m(X)$ holds whenever $WD_r^m(x)$ does for x running through all Zariski points of X and X is a variety over a field k (of characteristic distinct from l).
5. Conjecture $WD_r^n(X)$ holds whenever $n \geq r - 2$. This statement may be applied to $M = f_*(\mathbb{Q}_Y)$, where Y is a separated X -scheme possessing an X -compactification of δ -dimension at most r . In particular, for $B = X$ one may take $M = \mathbb{Q}_X$ whenever $\dim(X) \leq r$.
6. For X as in assertion 4 conjecture $WD_r^n(X)$ holds whenever $n \geq r - 2$ also.

Proof. 1, 2. Easy; note once again that $\text{Obj } DM_c(X)$ equals the union of $\widetilde{N}_r(X)\langle j \rangle$ as well as to the union of $\widetilde{N}'_r(X)\langle j \rangle$ for all $j \in \mathbb{Z}$ and $r \geq 0$.

3. Immediate from the DM -version of Theorem 3.2.5(4) (see Theorem 3.4.2(II)).

4. According to the previous assertion, it suffices to verify that the non-vanishing of $E_2^{pq}T_{w_{\text{Chow}}(X)}(H_{\mathbb{Q}_l}^{et}, M)$ (for some (p, q)) implies the non-vanishing of $E_2^{pq}T_{w_{\text{Chow}}(X)}(H, M)$ for H as in Conjecture 4.4.1(1).

Next, since any $M \in \text{Obj } DM_c(X)$ is defined over a subfield of k that is finitely generated over its prime subfield, it suffices to verify the statement for X of this form. Now (similarly to [Hub97]), one may use the seminal weight argument of Deligne and consider a “suitable model” of k , i.e., one should take a suitable variety over a finite field whose function field is k and shrink it to ensure the pointwise purity of the corresponding weight factors; this enables us to assume that k is a prime field itself.

Next, if k is finite then §5 of [BBD82] provides us with certain subcategories $D_c^b Sh^{et}(X, \mathbb{Q}_l)_{w \geq 0} \subset \text{Obj } D^c Sh^{et}(X, \mathbb{Q}_l)$ that are respected by the (idempotent) t -truncation functors $[-i] \circ H_i^t$. Hence the vanishing $E_2^{pq} T_{w_{Chow}(X)}(H_{\mathbb{Q}_l}^{et}, M)$ for all $q \in \mathbb{Z}$ and $p > n$ is equivalent to $\mathcal{H}_{\mathbb{Q}_l}^{et}(M) \in D_c^b Sh^{et}(X, \mathbb{Q}_l)_{w \geq 0}[-n]$. Since $D_c^b Sh^{et}(X, \mathbb{Q}_l)_{w \geq 0}$ is essentially defined in terms of $i_x^!$ (and pointwise weights), this allows us to conclude the proof in the case.

In the case when $k = \mathbb{Q}$ one should use the theory of weights developed in §3 of [Hub97] instead.

5. The first part of the assertion is immediate from assertion 3 combined with Proposition 4.2.3(4). One may apply it to X of this sort according to Theorem 3.4.2(II).

6. The usage of assertion 3 in the previous argument just should be replaced by that of assertion 4. □

Remark 4.4.3. 1. Take $Y = Y_N$ for $N \geq r + 1$ (see Proposition 4.3.5); then we have $\mathbb{Q}_X \notin DM_c(X)_{w_{Chow} \geq 1-r}$. Hence our theorem yields a non-vacuous relation between “étale weights” and negative K -groups in this “relative” setting also.

2. Now we discuss the applicability of our “perverse” criterion.

It was conjectured in §2.5 of [Bon15] that Chow-weight spectral sequences for perverse étale homology degenerate at E_2 (for any $M \in \text{Obj } DM_c(X)$) whenever they are defined. Thus in this case it suffices to study the factors of the corresponding Chow-weight filtrations.

3. Moreover, if X is a variety over a finite field, then Remark 3.6.2(1) of [Bon14] expresses the corresponding $E_2^{pq} T_{w_{Chow}(X)}(H_{\mathbb{Q}_l}^{et}, f_*(\mathbb{Q}_Y))$ as the q th weight factor of $R^{p+q} f_{*,et}(\mathbb{Q}_{lY})$ (cf. Proposition 4.3.1); here one should use the weights described in §5 of [BBD82]. Thus $c(Y/X) = n$ whenever $f_{*,et}(\mathbb{Q}_{lY}) \in D_c^b Sh^{et}(X, \mathbb{Q}_l)_{w \geq 0}[-n] \setminus D_c^b Sh^{et}(X, \mathbb{Q}_l)_{w \geq 0}[1-n]$ (see the proof of the theorem); in particular, one may apply this statement for $Y = X$ (and so, $f_{*,et}(\mathbb{Q}_{lY}) = q_X$). This seems to be (more or less) the only “relative” case where these E_2 -terms may be computed (at least, one may study their vanishing).

4. Note however that a certain theory of weights over number fields was developed in [Hub97]; one may apply it according to the results of [Bon14, §3.4]. More generally, one may apply the (more general) theory of weights developed in §2.5 of [Bon15] (along with the arguments used in the proof of Proposition 4.3.1; yet currently there does not exist a suitable theory of weights in the mixed characteristic case).

5. One can easily verify that the cohomology theory $H_x = H_{\mathbb{Q}_l}^{et}(x) \circ i_x^!$ and the weights spectral sequence $T_{w_{Chow}(X)}(H_x, M)$ can be described in terms of the “restriction” of M to any open neighbourhood of x in X . Hence for the purpose of computing $c(Y/X)$ it suffices to consider those point that belong to the image of the singular locus of Y .
6. One may apply the results proved above even in the case where there are no primes (“globally”) invertible on X . For this purpose one may invoke Proposition 2.3.1(9); it enables us to replace X by the components of its open cover such that on each of them invertible primes do exist (certainly, two components and two primes are sufficient here).

4.5 “Explicit test schemes” over varieties and the relation to the singularity filtration (on \mathcal{K} -theory and singular cohomology)

Now we relate our results to some (other) “descent” results and method. We start with a motivic of singularities results that it much simpler than the ones of §3 (and is related to the arguments used in [Bon11]).

Proposition 4.5.1. Let \mathcal{D} be as in Theorem 3.4.2(II) (so, it satisfies the local splitting condition) for B being the spectrum of a perfect field k of characteristic p .¹² Then the following statements are valid for $f: Y \rightarrow X$ being a separated morphism of (separated) finite type B -schemes, $d = \dim(Y)$.

1. $M = {}^{\mathcal{D}}\mathcal{M}_X^{BM}(Y)$ belongs to the envelope of ${}^{\mathcal{D}}\mathcal{M}_X^{BM}(Y_i)[-i]$ for some projective (separated) regular Y -schemes Y_i with $\dim(Y_i) \leq d - i$.
2. Assume that a full Karoubi-closed triangulated subcategory \underline{D} of $\mathcal{D}(X)$ is endowed with a weight structure w_X such that ${}^{\mathcal{D}}\mathcal{M}_X^{BM}(V)$ belongs to $\underline{D}_{w_X=0}$ for any regular Y that is proper over X .¹³ Then $M = {}^{\mathcal{D}}\mathcal{M}_X^{BM}(Y)$ belongs to $\text{Obj } \underline{D}$ and there exists a choice of $t_{w_X}(M) = (M^i)$ with $M^i = 0$ for $i < 0$ and $i > d$, and M_i is a retract of ${}^{\mathcal{D}}\mathcal{M}_X^{BM}(Y_i)[-i]$ for some projective (separated) regular X -scheme Y of dimension at most $d - i$ for $0 \leq i \leq d$.

Thus we have $M \in \underline{D}_{w_X \geq -n}$ for some non-negative integer n if and only if we have ${}^{\mathcal{D}}\mathcal{M}_X^{BM}(Y_i)[-i] \perp M$ for any $i \in \mathbb{Z}$ such that $n < i \leq d$ (and for these Y_i).

¹²Certainly, it follows that \mathcal{S} contains p whenever it is positive.

¹³So, the first two parts of this propositions may be applied for \mathcal{D} being equal either to $\mathcal{DK}(-)$ or to cdh -motives with R -coefficients, where R is any commutative unital ring in which p is invertible whenever it is positive; see Remark 2.1.3(2).

3. Now assume that $\mathcal{D}(-) = \mathcal{DK}(-)$ and f is proper. Then $M \in \mathcal{DK}(X)_{w_{\text{chow}} \geq -n}$ if and only if we have $\mathcal{K}_{-i}^{BM,X}(Y \times_X Y_i) = \{0\}$ for all $i > n$ and Y_i as above.
4. If $X = B(= \text{Spec } k)$ and k is perfect then the last (vanishing) condition is converted into $\mathcal{K}_{-i}(Y \times_X Y_i)$ being zero for all $i > n$.

Proof. **1.** Similarly to the proof of Proposition 3.2.4 (and according to Corollary 0.2 of [BoS15a]) we can assume that $\Lambda = \mathbb{Z}[\frac{1}{l}]$ for some prime $l \neq p$.

Next, applying the obvious induction on dimension argument we can assume that the assertion (for $\Lambda = \mathbb{Z}[\frac{1}{l}]$) is fulfilled whenever $\dim Y < d$. Now, according to Theorem X.2.1 of [ILO14], there exists a Cartesian square

$$\begin{array}{ccc}
 Z' & \xrightarrow{i'} & Y' \\
 \downarrow g' & & \downarrow g \\
 Z & \xrightarrow{i} & Y
 \end{array} \tag{4.2}$$

such that g is projective and generically finite of degree prime to l , i is a closed nowhere dense embedding, and Y' is regular. The local splitting property ensures that Z can be increased (still remaining closed and nowhere dense in Y) so that ${}^{\mathcal{D}}\mathcal{M}_X^{BM}(Y \setminus Z)$ will be a retract of ${}^{\mathcal{D}}\mathcal{M}_X^{BM}(Y' \setminus Z')$. Hence applying the ${}^{\mathcal{D}}\mathcal{M}^{BM}$ -version of (1.4) we obtain that ${}^{\mathcal{D}}\mathcal{M}_X^{BM}(Y)$ belongs to the envelope of $\{{}^{\mathcal{D}}\mathcal{M}^{BM}(Z), {}^{\mathcal{D}}\mathcal{M}^{BM}(Y'), {}^{\mathcal{D}}\mathcal{M}^{BM}(Z')[-1]\}$. Since Z and Z' are of dimension less than d , it remains to apply our inductive assumption to these schemes.

2. Consider an X -compactification \overline{Y} of Y (see Remark 3.1.3(2)); then $\dim(\overline{Y}) = d$ and $\dim(\overline{Y} \setminus Y) < d$. Hence applying the ${}^{\mathcal{D}}\mathcal{M}^{BM}$ -version of (1.4) once again we reduce the assertion to the case where f is proper. Hence the first part of the assertion follows from assertion **1** combined with Proposition 1.4.1(4). To deduce the second part from the first one it suffices to invoke (the “in particular” part of) Proposition 1.3.3(9).

3. Similarly to the proof of Theorem 2.2.1, we just use the isomorphism $DM(X)(\mathcal{MK}_X^{\dot{B}M}(Y_i)[-i], M) \cong \mathcal{K}_{-i}^{BM,X}(Y \times_X Y_i)$ given by (1.6).

4. Recall that the X -Borel–Moore \mathcal{K} -groups of smooth X -schemes are precisely the corresponding \mathcal{K} -groups; see (1.5) once again. \square

Remark 4.5.2. **1.** In all “reasonable” settings the weight structure w_X mentioned in part **2** of the proposition can be extended to the whole $\mathcal{D}(X)$; see Remark 3.3.3(3) of [Bon16a] for a result of this sort (though our current setting is much simpler than that of loc. cit.).

2. The advantage of the proposition is that its proof is quite simple, and one can more or less easily find the “detector varieties” Y_i (explicitly; note that the argument described in Remark 3.3.2(3) yields that it actually suffices to consider a finite number of primes when finding Y_i in the case of a general Λ).

On the other hand, it seems that to find “test schemes” for $M = f_*(\mathbf{1}_Y)$ in the case of a non-proper f one should use some more complicated arguments (similar to that described in §3).

3. Certainly our argument becomes somewhat simpler in the case where Hironaka’s resolution of singularities is available (so, for $p = 0$; cf. Remark 4.3.6(3)). The authors suspect that in this case the corresponding connected components of Y_i may be put into the “vertices” of the corresponding cubical diagram that yields a hyperresolution of Y ; see §2 of [PaP09].

Note moreover that in the current paper we use “lax” methods that allow us not to consider any enhancements for our motivic categories. This makes our arguments very general; yet they do not yield much information on the boundaries in $t_{w_X}(M)$ (cf. part 2 of the proposition). So it would make sense to consider certain “models” for our categories (that should be descent categories in the sense described in §2.1.3 of *ibid.*) and construct a Φ -rectified functor (see §2.1.5 of *ibid.*) from the category of cubical diagrams of finite type separated X -schemes into $\mathcal{DK}(X)$ (or into other motivic categories; for this purpose it actually suffices to construct a Φ -rectified functor into a descent category being a model for $SH(X)$). This would enable us to compare our weight spectral sequence $T_{w_{Chow}(X)}(\mathcal{DK}(X)(\mathbf{1}_X, -), \mathbf{1}_X)$ (along with the corresponding weight filtration) with the one given by Proposition 4.3 of *ibid.* (cf. also Corollary 5.9 of *ibid.*). Note here that the E_2 -descent property (see Proposition 5.6 of *ibid.*) is very easy to check for any cohomology that factors through one of our motivic categories: if (4.2) is an *elementary acyclic square* (i.e., g is an isomorphism over $Y \setminus Z$, and Z and Z' are also regular) then Proposition 1.3.3(6) yields that the corresponding distinguished triangle ${}^{\mathcal{D}}\mathbf{1}_Y \rightarrow {}^{\mathcal{D}}\mathcal{M}_Y^{BM}(Z) \oplus {}^{\mathcal{D}}\mathcal{M}_Y^{BM}(Y') \rightarrow {}^{\mathcal{D}}\mathcal{M}_Y^{BM}(Z')$ splits; hence we also have a split triangle ${}^{\mathcal{D}}\mathcal{M}_X^{BM}(Y) \rightarrow {}^{\mathcal{D}}\mathcal{M}_X^{BM}(Z) \oplus {}^{\mathcal{D}}\mathcal{M}_X^{BM}(Y') \rightarrow {}^{\mathcal{D}}\mathcal{M}_X^{BM}(Z')$ and thus a split exact sequence $0 \rightarrow H({}^{\mathcal{D}}\mathcal{M}_X^{BM}(Y)) \rightarrow H({}^{\mathcal{D}}\mathcal{M}_X^{BM}(Z)) \oplus H({}^{\mathcal{D}}\mathcal{M}_X^{BM}(Y')) \rightarrow H({}^{\mathcal{D}}\mathcal{M}_X^{BM}(Z')) \rightarrow 0$ for any additive functor H from $\mathcal{D}(X)$ (into an abelian category).

Similar arguments would probably allow to relate our results to [GiS96,

§5] and [GiS09] (note however that in these papers simplicial diagrams instead of cubical ones are used).

Another interesting spectral sequence of this sort (converging to the singular cohomology of X/\mathbb{C} , where X was allowed to be a complex analytic space) was considered in §5 of [CiG14]; it was called the *singularity spectral sequence* due to its triviality in the case of a regular X . Note that that this spectral sequence should be isomorphic to the Deligne’s weight one (starting from E_2) whenever X is proper over \mathbb{C} (see Corollary 5.3(2) of *ibid.* for the \mathbb{Q} -linear version of this statement) due to the fact that p_* is Chow-weight-exact whenever p is a proper morphism. The authors would like to note that the usage of Hironaka’s resolution of singularities is certainly a serious disadvantage of *ibid.* (along with [PaP09]) from the point of view of (general) algebraic geometry (since cubical hyperresolutions of singularities are available over characteristic 0 fields only). On the other hand, it would be very interesting to construct certain “complex analytic” weight structures related to the results of [CiG14].

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