EQUATIONS DETERMINING THE ORBIT OF THE HIGHEST WEIGHT VECTOR IN THE ADJOINT REPRESENTATION

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ABSTRACT. We explicitly construct a set of quadratic equations defining the highest weight vector orbit for adjoint representations of Chevalley groups of types D_l, E_6, E_7, and E_8. The combinatorics of these equations is related to the combinatorics of embeddings of the root system of type A_3. We believe that the constructed equations provide a prominent framework for calculations with exceptional groups in adjoint representations, which is particularly interesting for groups of type E_8.

1. INTRODUCTION

The highest weight vector orbit in an irreducible representation of a Chevalley group over an algebraically closed field is an intersection of quadrics (cf. [3]). We explicitly describe a set of quadratic equaions on this orbit over an arbitrary commutative ring. First of all, we have *square equations* described by Nikolai Vavilov in [7] for microweight representations as well as for adjoint ones. In some microweight cases those equations exhaust all equations defining the highest weight orbit (over an algebraically closed field). In the adjoint cases square equations are clearly not enough: for example (as Vavilov pointed out in [7]), they do not contain coordinates corresponding to the zero weight. We cannot get on with an A_2-proof of the structure theorems for E_8 (cf. [9]) without zero weight coordinates.

The equations on the highest weight vector orbit in the adjoint representation of a group of type A_l are well known: they are called *Plücker equations*. On the other hand, non-simply-laced root systems are generally a little harder to deal with. That is why we consider only the remaining simply-laced root systems D_l, E_6, E_7, and E_8. Moreover, in order to evade some difficulties relating to triality in D_4 we take $l \geq 5$ in the D_l case. In any way, we include D_l only because our constructions work verbatim in this case; our main goal is to obtain equations for exceptional groups.

We construct, in addition to the aforementioned square equations, two more classes of equations; all of them contain zero-weight coordinates. The combinatorics of these equations is also intimately related to the "numerology of maximal squares" studied in [7] and [8]. The same equations are produced in a more general context (but in slightly less explicit form) by Victor Petrov, Nikolai Vavilov, and myself in [4].

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2. The equations

Everywhere in this paper $\Phi = D_l$, $l \geq 5$ or $\Phi = E_l$, $l = 6, 7, 8$. Let $\{\alpha_1, \dots, \alpha_l\} = \Pi \subset \Phi$ be a fundamental system in Φ (its elements will be called fundamental roots). Our numbering of fundamental roots always follows Bourbaki [1]. For $\alpha \in \Phi$ we set $\alpha = \sum_{s=1}^{l} m_s(\alpha)\alpha_s$.

Let $G = G(\Phi, R)$ be the simply connected Chevalley group of type Φ over a commutative ring R with 1. We work with the adjoint representation of $G(\Phi, R)$, which gives us the irreducible action of $G(\Phi, R)$ on a free R-module V of rank $l(2l-1)$, 78, 133, 248 for $\Phi = D_l$, E_6, E_7, E_8 respectively. By Λ we denote the set of weights of our representation *with multiplicities*. More precisely, $\Lambda = \Lambda^* \sqcup \Delta$, where $\Lambda^* = \Phi$ is the set of non-zero weights, and $\Delta = \{0_1, \dots, 0_l\}$ is the set of zero weights. We fix an admissible base e^λ, $\lambda \in \Lambda$ in V. Hence we have the vectors e^α for $\alpha \in \Phi$ and $\widehat{e}^i = e^{0_i}$ for $i = 1, \dots, l$. Then a vector $v \in V$ can be uniquely written as $v = \sum_{\lambda \in \Lambda} v_\lambda e^\lambda = \sum_{\alpha \in \Phi} v_\alpha e^\alpha + \sum_{i=1}^{l} \widehat{v}_i \widehat{e}^i$. We will often simply write $v = (v_\lambda)$.

The root system Φ is a subset of a Euclidean space E with the scalar product denoted by $(\cdot,\cdot)$. We will also use a bilinear product defined by $\langle \alpha, \beta \rangle = 2(\alpha, \beta)/(\beta, \beta)$ for $\alpha, \beta \in E$ (for $\alpha, \beta \in \Phi$ we get the *Cartan numbers*). Note that our root system Φ is simply-laced, which means that all roots have length 1; therefore $\langle \alpha, \beta \rangle = 2(\alpha, \beta)$ for $\alpha, \beta \in \Phi$. We denote by $\angle(\alpha, \beta)$ the angle between $\alpha, \beta \in E$. Note that for $\alpha, \beta \in \Phi$ the scalar product (α, β) is $0, 1/2, -1/2, 1, -1$ if $\alpha \perp \beta$, $\alpha - \beta \in \Phi$, $\alpha + \beta \in \Phi$, $\alpha = \beta$, $\alpha = -\beta$ respectively.

The structure constants $N_{\alpha,\beta}$, $\alpha, \beta \in \Phi$ of the simple complex Lie algebra of type Φ are described in detail in [6, § 1]. We often use the identities for structure constants summarized there without any explicit reference. Note that in our case always $N_{\alpha,\beta} = 0$ or ± 1.

• **The $\pi/2$-equations.** Suppose $\alpha, \beta \in \Phi$ and $\angle(\alpha, \beta) = \pi/2$. Let us look at all other (unordered) pairs of roots with the same sum:

$$S_{\pi/2}(\alpha, \beta) = \{\{\gamma, \delta\} \mid \gamma + \delta = \alpha + \beta, \{\gamma, \delta\} \neq \{\alpha, \beta\}\}.$$

Consider the foolowing equation on a vector $v = (v_\lambda)_{\lambda \in \Lambda} \in V$:

$$v_\alpha v_\beta = \sum_{\{\gamma,\delta\} \in S_{\pi/2}(\alpha,\beta)} N_{\alpha,-\gamma} N_{\beta,-\delta} v_\gamma v_\delta. \tag{1}$$

We will call it *the $\pi/2$-equation* corresponding to the pair $\{\alpha, \beta\}$. First of all, we need to prove that the right hand side makes sense: we could swap γ with δ and get a different-looking coefficient. But it follows from the identity (C5) in [6] that $N_{\alpha,-\gamma} N_{\beta,-\delta} = N_{\alpha,-\delta} N_{\beta,-\gamma}$. Next, note that $(\alpha, \gamma) + (\alpha, \delta) = (\alpha, \gamma + \delta) = (\alpha, \alpha + \beta) = (\alpha, \alpha) = 1$, while $\alpha \neq \gamma$, $\alpha \neq \delta$. Therefore $(\alpha, \gamma) = (\alpha, \delta) = 1/2$. It follows that $\angle(\alpha, \gamma) = \angle(\alpha, \delta) = \pi/3$.

For the rest of the paper, put $k = l, 4, 5, 7$ for $\Phi = D_l, E_6, E_7, E_8$, respectively. In order to write the $\pi/2$-equation in a more symmetric form, recall a definition from [7].

Definition 1. *A set of roots $\{\beta_i\}$, $i = 1, \dots, k, -k, \dots, -1$ such that $\angle(\beta_i, \beta_{-i}) = \pi/2$ for every $i = 1, \dots, k$, and $\angle(\beta_i, \beta_j) = \pi/3$ for $i \neq \pm j$, is called a* **maximal square**.

For a maximal square $\{\beta_i\}$ the sum $\beta_i + \beta_{-i}$ does not depend on i. Therefore the set of roots contained in the pairs from $S_{\pi/2}(\alpha, \beta)$, together with the roots α and β, is a maximal square (this was proved in [7, Theorem 1]). We shall prove shortly that our $\pi/2$-equation is uniquely determined by this maximal square, and does not depend on the choice of an orthogonal pair of roots $\{\alpha, \beta\}$. Let us fix an index $j = 1, \dots, -1$. If we put $\beta_1 = \alpha$, $\beta_{-1} = \beta$, and $S_{\pi/2}(\alpha, \beta) = \{\{\beta_i, \beta_{-i}\} \mid i = 2, \dots, k\}$, the $\pi/2$-equation can be rewritten as

$$\nu_{\beta_1}\nu_{\beta_{-1}} = \sum_{i\geq 2} N_{\beta_1,-\beta_i} N_{\beta_{-1},-\beta_{-i}} \nu_{\beta_i}\nu_{\beta_{-i}}.$$

The sign column $c(j) \in (\mathbb{Z}/2\mathbb{Z})^{2k}$ is defined as follows.

$$c(j)_i = \begin{cases} 1, & \text{if } i = \pm j, \\ -N_{\beta_j,-\beta_i} N_{\beta_{-j},-\beta_{-i}}, & \text{if } i \neq \pm j. \end{cases}$$

Another equivalent form of the $\pi/2$-equation is

$$\sum_{i=1}^{k} c(1)_i \nu_{\beta_i}\nu_{\beta_{-i}} = 0.$$

The following lemma says that if we take another orthogonal pair in $S_{\pi/2}(\alpha, \beta)$ instead of $\{\alpha, \beta\}$, we will get the same equation.

Lemma 1. *For any* $j, h = 1, \dots, -1$ *we have*

$$c(h) - c(h)_j c(j) = 0.$$

Proof. Immediately follows from [7, Theorem 3]. □

• **The** $2\pi/3$**-equations.**

Suppose again that $\alpha, \beta \in \Phi$ and $\angle(\alpha, \beta) = \pi/2$. Consider all pairs of roots $\{\gamma, \delta\}$ such that $\gamma + \delta = \alpha$ and γ, δ are not orthogonal to β. Note that if $\gamma \perp \beta$ and $\gamma + \delta = \alpha$, then $(\delta, \beta) = (\alpha - \gamma, \beta) = 0$, so $\delta \perp \beta$. Also, $0 = (\alpha, \beta) = (\gamma + \delta, \beta) = (\gamma, \beta) + (\delta, \beta)$. Therefore for such a pair $\{\gamma, \delta\}$ one of the angles $\angle(\gamma, \beta)$, $\angle(\delta, \beta)$ is $2\pi/3$, while the other is $\pi/3$. Put

$$S_{2\pi/3}(\alpha, \beta) = \{\{\gamma, \delta\} \mid \gamma + \delta = \alpha, (\gamma, \beta) \neq 0\}.$$

Consider the following equation on a vector $\nu = (\nu_\lambda)_{\lambda \in \Lambda} \in V$:

$$\nu_\alpha \cdot \sum_{s=1}^{l} \langle \beta, \alpha_s \rangle \widehat{\nu}_s = - \sum_{\substack{\{\gamma,\delta\} \in S_{2\pi/3}(\alpha,\beta), \\ \angle(\gamma,\beta) = \pi/3}} N_{\gamma,\delta} \nu_\gamma \nu_\delta. \tag{2}$$

We will call it *the* $2\pi/3$*-equation* corresponding to the pair (α, β).

The pairs in $S_{2\pi/3}(\alpha, \beta)$ are related to the embeddings of root systems $A_3 \subset \Phi$. In order to see that, consider a pair $\{\gamma, \delta\} \in S_{2\pi/3}(\alpha, \beta)$. We may assume that $(\gamma, \beta) = 1/2$, $(\delta, \beta) = -1/2$. Then the roots $\delta, \gamma, \beta - \gamma$ form a fundamental system of a root subsystem $\Psi \subseteq \Phi$ of type A_3. We can write the roots α, β in the Dynkin notation for this fundamental system as follows: $\alpha = 110$, $\beta = 011$. Note that Ψ contains γ', δ' for exactly one more pair $\{\gamma', \delta'\} \in S_{2\pi/3}(\alpha, \beta)$, namely, the pair $\{\gamma', \delta'\} = \{111, -001\}$. In other words, $\gamma' = \delta + \beta$, $\delta' = \gamma - \beta$. The pairs $\{\gamma, \delta\}$ and $\{\gamma', \delta'\}$ are said to be*conjugate*.

Note that $|S_{2\pi/3}(\alpha, \beta)| = 2(l-1), 6, 8, 12$ for $\Phi = D_l, E_6, E_7, E_8$, respectively. We see that the number of conjugate pairs in $S_{2\pi/3}(\alpha, \beta)$ is one less than the number of pairs of orthogonal roots in a maximal square. This is not a coincidence: if we fix an orthogonal pair (α, β) in a maximal square and take any of the remaining pairs, together they span a root subsystem of type A_3. There are exactly $k - 1$ of these subsystems, and each contains exactly two of conjugate pairs from $S_{2\pi/3}(\alpha, \beta)$.

We get the following equivalent description of $S_{2\pi/3}(\alpha, \beta)$:

Lemma 2. *Suppose that $\alpha, \beta \in \Phi$, $\alpha \perp \beta$. Let $\Omega = \{\beta_1, \dots, \beta_{-1}\}$ be a maximal square such that $\beta_1 = \alpha$, $\beta_{-1} = \beta$, and $\beta_i \perp \beta_{-i}$ for every i. Then*

$$S_{2\pi/3}(\alpha, \beta) = \{\{\beta_1 - \beta_i, \beta_i\} \mid i = 2, \dots, -2\},$$

and the $2\pi/3$-equation corresponding to the pair (α, β) can be rewritten as follows:

$$v_{\beta_1} \cdot \sum_{s=1}^{l} \langle \beta_{-1}, \alpha_s \rangle \widehat{v}_s = \sum_{i \neq \pm 1} N_{\beta_1, -\beta_i} v_{\beta_1 - \beta_i} v_{\beta_i}.$$

Proof. Note that $(\beta_1 - \beta_i) + \beta_i = \beta_1 = \alpha$ and $(\beta_i, \beta) = (\beta_i, \beta_{-1}) \neq 0$ for $i = 2, \dots, -2$. This means that all pairs $\{\beta_1 - \beta_i, \beta_i\}$ are in $S_{2\pi/3}(\alpha, \beta)$. In order to prove the reverse inclusion, consider a pair $\{\gamma, \delta\} \in S_{2\pi/3}(\alpha, \beta)$. We may assume that $\angle(\gamma, \beta) = 2\pi/3$. Then the roots $\alpha - \gamma$, $\gamma + \beta$ are orthogonal, and their sum is $\alpha + \beta$; it follows that $\alpha - \gamma, \gamma + \beta \in \Omega$, so that $\delta = \alpha - \gamma = \beta_i$ for some i. It remains to note that $N_{\beta_1 - \beta_i, \beta_i} = N_{\beta_i, -\beta_1} = N_{\beta_1, -\beta_i}$ by the identities (C4) and (C1) from [6] □

• **The π-equations.** Suppose that $\alpha, \beta \in \Phi$ and $\angle(\alpha, \beta) = \pi/2$. Consider all pairs of roots $\{\gamma, \delta\}$ such that $\gamma = -\delta$ and γ, δ are not orthogonal to α and β. There are two possibilities: the first is $(\gamma, \alpha) = (\gamma, \beta)$, and then $(\delta, \alpha) = (\delta, \beta)$. We may assume that $(\gamma, \alpha) = (\gamma, \beta) = 2\pi/3$. Put

$$S_\pi(\alpha, \beta) = \{(\gamma, \delta) \mid \gamma + \delta = 0, \angle(\gamma, \alpha) = \angle(\gamma, \beta) = 2\pi/3\}.$$

The second possibility is that one of the angles $\angle(\gamma, \alpha)$, $\angle(\delta, \alpha)$ is $2\pi/3$. We may assume that $\angle(\gamma, \alpha) = 2\pi/3$, and then $\angle(\gamma, \beta) = \pi/3$, $\angle(\delta, \alpha) = \pi/3$, $\angle(\delta, \beta) = 2\pi/3$. Put

$$S'_\pi(\alpha, \beta) = \{(\gamma, \delta) \mid \gamma + \delta = 0, \angle(\gamma, \alpha) = \angle(\delta, \beta) = 2\pi/3\}.$$

Consider the following equation on a vector $v = (v_\lambda)_{\lambda \in \Lambda} \in V$:

$$\sum_{s=1}^{l} \langle \alpha, \alpha_s \rangle \widehat{v}_s \cdot \sum_{s=1}^{l} \langle \beta, \alpha_s \rangle \widehat{v}_s = \sum_{(\gamma,\delta) \in S'_\pi(\alpha,\beta)} v_\gamma v_\delta - \sum_{(\gamma,\delta) \in S_\pi(\alpha,\beta)} v_\gamma v_\delta. \tag{3}$$

We will call it *the π-equation* corresponding to the pair (α, β).

Note that $|S_\pi(\alpha, \beta)| = |S'_\pi(\alpha, \beta)| = 2(l-1), 6, 8, 12$ for $\Phi = D_l, E_6, E_7, E_8$, respectively. As in the previous case, we can construct a maximal square corresponding to $S_\pi(\alpha, \beta)$. For any pair $(\gamma, \delta) \in S_\pi(\alpha, \beta)$ we have $\gamma + \alpha \in \Phi$ $(\gamma + \alpha, \beta) = (\gamma, \beta) = -1/2$, therefore $\gamma + \alpha + \beta \in \Phi$. Moreover, $(\gamma + \alpha + \beta, \alpha) = (\gamma + \alpha + \beta, \beta) = (-\gamma, \alpha) = (-\gamma, \beta) = 1/2$ and $-\gamma + (\gamma + \alpha + \beta) = \alpha + \beta$. This means that the roots $\{-\gamma \mid (\gamma, \delta) \in S_\pi(\alpha, \beta)\}$ together with α, β form a maximal square. It is easy to see that the roots $\{\gamma + \alpha, \beta + \delta \mid (\gamma, \delta) \in S'_\pi(\alpha, \beta)\}$ together with α, β form (the same) maximal square.

As in the previous case, the constructed sets $S_\pi(\alpha, \beta)$ and $S'_\pi(\alpha, \beta)$ are related to embeddings $A_3 \subset \Phi$: if $(\gamma, \delta) \in S_\pi(\alpha, \beta)$, the roots α, γ, β form a fundamental system of a root subsystem $\Psi \subseteq \Phi$ of type A_3. We can write $(\gamma, \delta) = (010, -010)$ in Dynkin notation with respect to this fundamental system. Moreover, Ψ contains another pair of roots from $S_\pi(\alpha, \beta)$, namely, $(-111, 111)$. On the other hand, the pairs $(-110, 110)$ and $(011, -011)$ are in $S'(\alpha, \beta)$. The analogue of Lemma 2 holds in this situation:

Lemma 3. *Suppose that $\alpha, \beta \in \Phi$, $\alpha \perp \beta$. Let $\Omega = \{\beta_1, \dots, \beta_{-1}\}$ be a maximal square such that $\beta_1 = \alpha$, $\beta_{-1} = \beta$, and $\beta_i \perp \beta_{-i}$ for every i. Then*

$$S_\pi(\alpha, \beta) = \{(-\beta_i, \beta_i) \mid i = 2, \dots, -2\},$$
$$S'_\pi(\alpha, \beta) = \{(\beta_i - \beta_1, \beta_1 - \beta_i) \mid i = 2, \dots, -2\},$$

and the π-equation corresponding to (α, β) can be rewritten is follows.

$$\sum_{s=1}^{l}\langle\beta_1, \alpha_s\rangle\widehat{\nu}_s \cdot \sum_{s=1}^{l}\langle\beta_{-1}, \alpha_s\rangle\widehat{\nu}_s = \sum_{i\neq\pm1}(\nu_{\beta_1-\beta_i}\nu_{\beta_i-\beta_1} - \nu_{-\beta_i}\nu_{\beta_i}).$$

To reiterate, we get one $\pi/2$-equation for every maximal square, one $2\pi/3$-equation and one π-equation for every maximal square with a chosen pair of orthogonal roots in it.

3. Preliminary lemmas

We encountered embeddings $A_3 \subseteq \Phi$; we will use the fact that every such embedding can be expanded to an embedding $D_4 \subseteq \Phi$.

Lemma 4. *Recall that $\Phi = E_l$ or D_l ($l \geq 5$). Every subsystem $\Psi \subseteq \Phi$ of type A_3 can be embedded into a subsystem of type D_4. To be precise, if $\alpha, \beta, \gamma \in \Phi$ are roots such that $\alpha \perp \gamma$, $\angle(\alpha, \beta) = \angle(\beta, \gamma) = 2\pi/3$, then there is a root $\delta \in \Phi$ such that $\delta \perp \alpha$, $\delta \perp \gamma$, and $\angle(\delta, \beta) = 2\pi/3$.*

Proof. In the case $\Phi = E_l$ all subsystems of type A_3 in Φ lie in one orbit with respect to the action of the Weyl group $W(E_l)$. This follows, for example, from the tables in Carter's paper [2]. Therefore it remains to show the statement for a single subsystem of type A_3: for example, we may assume that $\alpha = \alpha_2$, $\beta = \alpha_4$, $\gamma = \alpha_3$ and take $\delta = \alpha_5$. In the case $\Phi = D_l$ there are two orbits of subsystems of type A_3 with respect to the action of the Weyl group $W(D_l)$. This immediately follows from the computations in [2, § 9]. For one of the orbits we may assume that $\alpha = \alpha_{l-1}$, $\beta = \alpha_{l-2}$, $\gamma = \alpha_l$, and take $\delta = \alpha_{l-3}$; for the other orbit we may assume that $\alpha = \alpha_{l-3}$, $\beta = \alpha_{l-2}$, $\gamma = \alpha_{l-1}$, and take $\delta = \alpha_l$. ☐

Now we describe the possible relative positions of a root $\rho \in \Phi$ and a maximal square $\Omega = \{\beta_1, \dots, \beta_{-1}\}$.

Lemma 5. *Let $\Omega = \{\beta_1, \dots, \beta_{-1}\}$ be a maximal square, and let $\rho \in \Phi$ be a root. Exactly one of the following holds:*

(1) *There exists i such that $\rho = \beta_i$, $\angle(\rho, \beta_{-i}) = \pi/2$, and $\angle(\rho, \beta_j) = \pi/3$ for $j \neq \pm i$.*

(2) *There exists i such that $\rho = -\beta_i$, $\angle(\rho, \beta_{-i}) = \pi/2$, and $\angle(\rho, \beta_j) = 2\pi/3$ for $j \neq \pm i$.*

(3) *There exists i such that $\rho \perp \beta_i$ and $\rho \perp \beta_{-i}$; for every $j = 1, \dots, -1$ either $\rho \perp \beta_j$, $\rho \perp \beta_{-j}$, or one of the angles $\angle(\rho, \beta_j)$, $\angle(\rho, \beta_{-j})$ equals $\pi/3$, while the other equals $2\pi/3$.*

(4) *For every i one of the angles $\angle(\rho, \beta_i)$, $\angle(\rho, \beta_{-i})$ equals $\pi/2$, while the other equals $\pi/3$.*

(5) *For every i one of the angles $\angle(\rho, \beta_i)$, $\angle(\rho, \beta_{-i})$ equals $\pi/2$, while the other equals $2\pi/3$.*

Moreover, $(\rho, \beta_i + \beta_{-i})$ equals 1, -1, 0, $1/2$, $-1/2$ in cases (1), (2), (3), (4), (5), respectively.

Proof. If $\rho \in \Omega$, we have $\rho = \beta_i$ for some i. By the definition of maximal square, the angle between ρ and every other root in Ω is equal to $\pi/3$. In this case, (1) holds.

If $-\rho \in \Omega$, we can apply the above observation to $-\rho$; in this case, (2) holds.

Now we assume that $\pm\rho \notin \Omega$. Hence, (ρ, β_i) is equal to 0 or $\pm 1/2$. Suppose that there exists i such that $(\rho, \beta_i) = 0$.

- If there exists i such that $(\rho, \beta_i) = (\rho, \beta_{-i}) = 0$, then for every $j \neq i$ we have

$$(\rho, \beta_j) + (\rho, \beta_{-j}) = (\rho, \beta_j + \beta_{-j}) = (\rho, \beta_i + \beta_{-i}) = 0.$$

This means that either $(\rho, \beta_j) = (\rho, \beta_{-j}) = 0$, or one of these scalar products equals $1/2$, and the other equals $-1/2$. Therefore (3) holds. Note that in this case there exists j such that $(\rho, \beta_j) \neq 0$: otherwise ρ would be orthogonal to every root in Ω, which is impossible.

- If there exists i such that $(\rho, \beta_i) = 0$ and $(\rho, \beta_{-i}) = 1/2$, then for every $j \neq i$ we have $(\rho, \beta_j) + (\rho, \beta_{-j}) = 1/2$. This means that one of these products is equal to 0, and the other is equal to $1/2$. Therefore (4) holds.
- Similarly, if there exists i such that $(\rho, \beta_i) = 0$ and $(\rho, \beta_{-i}) = -1/2$, then for every $j \neq i$ we have $(\rho, \beta_j) + (\rho, \beta_{-j}) = -1/2$. This means that one of these products is equal to 0, and the other is equal to $-1/2$. Therefore (4) holds.

Finally let us consider the remaining case: suppose that for every i the scalar product (ρ, β_i) is not equal to 0. We must show that this is impossible. If for some i we have $(\rho, \beta_i) = (\rho, \beta_{-i}) = 1/2$, then $\beta_i + \beta_{-i} - \rho$ is a root, and its sum with ρ is $\beta_i + \beta_{-i}$. Therefore $\rho \in \Omega$, and we are in the case (1).

On the other hand, if for some i we have $(\rho, \beta_i) = (\rho, \beta_{-i}) = -1/2$, then $\rho + \beta_i + \beta_{-i} \in \Phi$ and $-\rho + (\rho + \beta_i + \beta_{-i}) = \beta_i + \beta_{-i}$. Therefore $-\rho \in \Omega$, and we are in the case (2).

Finally, we can choose i such that $(\rho, \beta_i) = 1/2$, $(\rho, \beta_{-i}) = -1/2$. Let us show that we are in the case (3). The roots $-\beta_i, \rho, \beta_{-i}$ span a root subsystem $\Psi \subseteq \Phi$ of type A_3. By Lemma 4, we can embed it into a root subsystem of type D_4. Therefore there exists $\sigma \in \Phi$ such that $\sigma \perp \beta_i$, $\sigma \perp \beta_{-i}$, and $(\sigma, \rho) = -1/2$. But $-\sigma - \rho + \beta_i, \sigma + \rho + \beta_{-i} \in \Phi$, the sum of these two roots is $\beta_i + \beta_{-i}$, and both of them are orthogonal to ρ. This means that we are in the case (3). □

Definition 2. *Let Ω be a maximal square in Φ, and let $\rho \in \Phi$ be a root. We say that the angle between ρ and Ω is equal to $\angle(\rho, \Omega) = 0, \pi, \pi/2, \pi/3, 2\pi/3$, if in the Lemma 5 the condition* (1), (2), (3), (4), (5) *holds, respectively.*

Lemma 6. *Let $\Omega = \{\beta_1, \dots, \beta_{-1}\}$ be a maximal square. Suppose that $j \in \{1, \dots, -1\}$. Put $\gamma_i = \beta_j - \beta_i$ for every $i \neq \pm j$, $\gamma_j = \beta_j$, $\gamma_{-j} = -\beta_{-j}$. $\Omega' = \{\gamma_1, \dots, \gamma_{-1}\}$ is a maximal square.*

Proof. An easy calculation shows that $(\gamma_i, \gamma_{-i}) = 0$ for all i, and $(\gamma_i, \gamma_t) = 1/2$ for all $i \neq t$. □

4. Action of the elementary subgroup

Suppose $\rho \in \Phi$, $\xi \in R$. We work with the adjoint representation, therefore the action of the elementary root unipotent $x_\rho(\xi)$ on the basis of V is described by the following lemma.

Lemma 7 (Matsumoto). *(1) If $\lambda \in \Phi$, $\lambda + \rho \notin \Phi \cup \{0\}$, then $x_\rho(\xi)e^\lambda = e^\lambda$;*
(2) if $\lambda, \lambda + \rho \in \Phi$, then $x_\rho(\xi)e^\lambda = e^\lambda + N_{\rho,\lambda}\xi e^{\lambda+\rho}$;
(3) $x_\rho(\xi)\widehat{e}^s = \widehat{e}^s - \langle \rho, \alpha_s \rangle \xi e^\rho$ for $s = 1, \dots, l$;
(4) $x_\rho(\xi)e^{-\rho} = e^{-\rho} + \sum_{s=1}^{l} m_s(\rho)\xi\widehat{e}^s - \xi^2 e^\rho$.

Proof. See [5, Lemma 2.3]. □

This immediately implies the following description of the action of $x_\rho(\xi)$ on the coordinates of $v = (v_\lambda) \in V$. We will use it without any further reference.

Lemma 8. *(1) If $\lambda \in \Phi$, $\lambda - \rho \notin \Phi \cup \{0\}$, then $(x_\rho(\xi)v)_\lambda = v_\lambda$;*
(2) if $\lambda, \lambda - \rho \in \Phi$, then $(x_\rho(\xi)v)_\lambda = v_\lambda + N_{\rho,\lambda-\rho}\xi v_{\lambda-\rho}$;
(3) $\widehat{(x_\rho(\xi)v)}_s = \widehat{v}_s + m_s(\rho)\xi v_{-\rho}$.

(4) $(x_\rho(\xi)v)_\rho = v_\rho - \sum_{s=1}^{l}\langle\rho,\alpha_s\rangle\xi\widehat{v}_s - \xi^2 v_{-\rho}$.
In particular, if $\angle(\rho,\lambda) = \pi/2$, $2\pi/3$ *or* π, *then* $(x_\rho(\xi)v)_\lambda = v_\lambda$.

We will often use the following description of the action of $x_\rho(\xi)$ on the zero weights.

Lemma 9. *Suppose that* $\beta,\rho\in\Phi$, $\xi\in R$, $v\in V$, $w = x_\rho(\xi)v$. *Then* $\sum_s\langle\beta,\alpha_s\rangle\widehat{w}_s = \sum_s\langle\beta,\alpha_s\rangle\widehat{v}_s + \xi\langle\beta,\rho\rangle v_{-\rho}$.

Proof.

$$\begin{aligned}\sum_s\langle\beta,\alpha_s\rangle\widehat{w}_s &= \sum_s\langle\beta,\alpha_s\rangle(\widehat{v}_s + m_s(\rho)\xi v_{-\rho})\\ &= \sum_s\langle\beta,\alpha_s\rangle\widehat{v}_s + \xi\sum_s\langle\beta, m_s(\rho)\alpha_s\rangle v_{-\rho}\\ &= \sum_s\langle\beta,\alpha_s\rangle\widehat{v}_s + \xi\langle\beta,\rho\rangle v_{-\rho}\end{aligned}$$

□

Suppose that $v\in V$, $\alpha,\beta\in\Phi$, and $\Omega = \{\beta_1,\dots,\beta_{-1}\}$ is a maximal square such that $\beta_1 = \alpha$, $\beta_{-1} = \beta$, and $\beta_i\perp\beta_{-i}$ for every i. We need the following notation for the polynomials in the equations (1), (2), (3):

$$f^{\pi/2}_{\alpha,\beta}(v) = v_\alpha v_\beta - \sum_{\{\gamma,\delta\}\in S_{\pi/2}(\alpha,\beta)} N_{\alpha,-\gamma}N_{\beta,-\delta}v_\gamma v_\delta,$$

$$f^{2\pi/3}_{\alpha,\beta}(v) = \sum_{i\neq\pm1} N_{\beta_1,-\beta_i}v_{\beta_1-\beta_i}v_{\beta_i} - v_{\beta_1}\sum_{s=1}^{l}\langle\beta_{-1},\alpha_s\rangle\widehat{v}_s,$$

$$f^{\pi}_{\alpha,\beta}(v) = \sum_{i\neq\pm1}(v_{\beta_1-\beta_i}v_{\beta_i-\beta_1} - v_{-\beta_i}v_{\beta_i}) - \sum_{s=1}^{l}\langle\beta_1,\alpha_s\rangle\widehat{v}_s\cdot\sum_{s=1}^{l}\langle\beta_{-1},\alpha_s\rangle\widehat{v}_s.$$

Proposition 1. *Let* $\alpha,\beta,\rho\in\Phi$ *be roots such that* $\alpha\perp\beta$, *and let* $v\in V$ *be a vector. Take* $\xi\in R$ *and put* $w = x_\rho(\xi)v$. *Suppose that* $\varphi\in\{\pi/2, 2\pi/3,\pi\}$. *Then* $f^{\varphi}_{\alpha,\beta}(w)$ *is a linear combination of polynomials of the form* $f^{\psi}_{\gamma,\delta}(v)$.

We shall prove Proposition 1 in the next section. Now we can derive our main result from it.

Theorem 1. *The set of vectors* $v\in V$ *satisfying the equations* (1), (2), (3) *for all* $\alpha,\beta\in\Phi$, $\alpha\perp\beta$, *is invariant under the action of the group* $E(\Phi,R)$.

Proof. It suffices to prove that if $v\in V$ satisfies the above equations, then $w = x_\rho(\xi)v$ satisfies them for every $\rho\in\Phi$, $\xi\in R$. Indeed, by Proposition 1, each of the polynomials $f^{\varphi}_{\alpha,\beta}(w)$ is equal to a linear combination of these polynomials applied to v, which is zero. □

Corollary 1. *If* $v\in V$ *is a column of an element* $g\in E(\Phi,R)$ *corresponding to any root* $\rho\in\Lambda^* = \Phi$, *then* v *satisfies the equations* (1), (2), (3) *for all* $\alpha,\beta\in\Phi$, $\alpha\perp\beta$.

Proof. We have $v = ge^\rho$. It is obvious that e^ρ satisfies those equations, so by Theorem 1 v satisfies them too. □

5. Proof of Proposition 1

Let $\Omega = \{\beta_1,\dots,\beta_{-1}\}$ is a maximal square such that $\beta_1 = \alpha$, $\beta_{-1} = \beta$, and $\beta_i\perp\beta_{-i}$ for every i. We shall explore the five cases described in Lemma 5.

(1) Suppose that $(\rho,\Omega) = 0$. This means that $\rho = \beta_j$ for some j.

- **The $\pi/2$-equation.** The discussion following Definition 1 shows that the $\pi/2$-equation depends only on a maximal square and not on the choice of orthogonal roots α, β in it. Thus we may assume that $j = 1$. Then

$$
\begin{aligned}
&f^{\pi/2}_{\alpha,\beta}(w) - f^{\pi/2}_{\alpha,\beta}(v)\\
&= \left(-\sum_{s=1}^{l}\langle\beta_1,\alpha_s\rangle\xi\widehat{v}_s - \xi^2 v_{-\beta_1}\right) v_{\beta_{-1}}\\
&- \xi\sum_{i\neq\pm1} N_{\beta_1,-\beta_i}N_{\beta_{-1},-\beta_{-i}}N_{\beta_1,\beta_i-\beta_1}v_{\beta_i-\beta_1}v_{\beta_{-i}}\\
&- \xi^2\sum_{i\geq 2} N_{\beta_1,-\beta_i}N_{\beta_{-1},-\beta_{-i}}N_{\beta_1,\beta_i-\beta_1}N_{\beta_1,\beta_{-i}-\beta_1}v_{\beta_i-\beta_1}v_{\beta_{-i}-\beta_1}.
\end{aligned}
$$

First, note that $N_{\beta_1,\beta_i-\beta_1} = -N_{\beta_1,-\beta_i}$. Moreover, $N_{\beta_{-1},-\beta_{-i}} = N_{\beta_{-1},\beta_1-\beta_i}$ and $N_{\beta_1,\beta_{-i}-\beta_1} = -N_{-\beta_1,\beta_1-\beta_{-i}}$. Therefore the terms on the right-hand side containing ξ^2 sum up to $-\xi^2 f^{\pi/2}_{\beta_{-1},-\beta_1}(v)$. The rest sums up to $-\xi f^{2\pi/3}_{\beta_{-1},-\beta_1}(v)$. Indeed, $N_{\beta_1,-\beta_i} = -N_{\beta_1,\beta_i-\beta_1}$, and (by Lemma 6) the roots $\beta_i - \beta_1$ together with β_{-1} and $-\beta_1$ form a maximal square. Hence

$$f^{\pi/2}_{\alpha,\beta}(w) - f^{\pi/2}_{\alpha,\beta}(v) = -\xi f^{2\pi/3}_{\beta_{-1},-\beta_1}(v) - \xi^2 f^{\pi/2}_{\beta_{-1},-\beta_1}(v).$$

- **The $2\pi/3$-equation.** First suppose that $j \neq \pm1$. Then $w_{\beta_1-\beta_i} = v_{\beta_1-\beta_i}$ for $i \neq -j$. Thus

$$
\begin{aligned}
f^{2\pi/3}_{\alpha,\beta}(w) - f^{2\pi/3}_{\alpha,\beta}(v) = &\sum_{\substack{i\neq\pm1\\ i\neq\pm j}}\left(N_{\beta_1,-\beta_i}v_{\beta_1-\beta_i}\xi N_{\beta_j,\beta_i-\beta_j}v_{\beta_i-\beta_j}\right)\\
&+ N_{\beta_1,-\beta_{-j}}\xi N_{\beta_j,-\beta_{-1}}v_{-\beta_{-1}}v_{\beta_{-j}}\\
&- N_{\beta_1,-\beta_j}v_{\beta_1-\beta_j}\sum_{s=1}^{l}\langle\beta_j,\alpha_s\rangle\xi\widehat{v}_s\\
&- N_{\beta_1,-\beta_j}v_{\beta_1-\beta_j}\xi^2 v_{-\beta_j}\\
&- \xi N_{\beta_j,\beta_1-\beta_j}v_{\beta_1-\beta_j}\sum_{s=1}^{l}\langle\beta_{-1},\alpha_s\rangle\widehat{v}_s\\
&- v_{\beta_1}\xi\langle\beta_{-1},\beta_j\rangle v_{-\beta_j}\\
&- \xi^2 N_{\beta_j,\beta_1-\beta_j}v_{\beta_1-\beta_j}\langle\beta_{-1},\beta_j\rangle v_{-\beta_j}.
\end{aligned}
$$

Note that $\langle\beta_{-1},\beta_j\rangle = 1$ and $N_{\beta_j,\beta_1-\beta_j} = -N_{\beta_1,-\beta_j}$, so the terms containing ξ^2 cancel each other out. Using the cocycle identity

$$N_{\beta_1,-\beta_i}N_{\beta_j-\beta_1,\beta_1-\beta_i} = N_{\beta_j,-\beta_i}N_{\beta_j-\beta_1,\beta_1}$$

and Lemma 6, it is easy to show that the rest yields

$$f^{2\pi/3}_{\alpha,\beta}(w) - f^{2\pi/3}_{\alpha,\beta}(v) = \xi N_{\beta_1,-\beta_j}f^{2\pi/3}_{\beta_1-\beta_j,\beta_1-\beta_{-j}}(v).$$

Now suppose that $j = 1$. By Lemma 9 we have $\sum_{s=1}^{l} \langle \beta_{-1}, \alpha_s \rangle \widehat{w}_s = \sum_{s=1}^{l} \langle \beta_{-1}, \alpha_s \rangle \widehat{v}_s$, since $\langle \beta_{-1}, \beta_1 \rangle = 0$. Therefore

$$\begin{aligned} f^{2\pi/3}_{\alpha,\beta}(w) - f^{2\pi/3}_{\alpha,\beta}(v) = &\sum_{i \neq \pm 1} N_{\beta_1,-\beta_i} \xi N_{\beta_1,-\beta_i} v_{-\beta_i} v_{\beta_i} \\ &+ \sum_{i \neq \pm 1} N_{\beta_1,-\beta_i} v_{\beta_1-\beta_i} \xi N_{\beta_1,\beta_i-\beta_1} v_{\beta_i-\beta_1} \\ &+ \sum_{i \neq \pm 1} N_{\beta_1,-\beta_i} \xi^2 N_{\beta_1,-\beta_i} N_{\beta_1,\beta_i-\beta_1} v_{-\beta_i} v_{\beta_i-\beta_1} \\ &+ \left(\sum_{s=1}^{l} \langle \beta_1, \alpha_s \rangle \xi \widehat{v}_s + \xi^2 v_{-\beta_1} \right) \sum_{s=1}^{l} \langle \beta_{-1}, \alpha_s \rangle \widehat{v}_s . \end{aligned}$$

Note that $N_{\beta_1,\beta_i-\beta_1} = -N_{\beta_1,-\beta_i}$. It is easy to see that

$$f^{2\pi/3}_{\alpha,\beta}(w) - f^{2\pi/3}_{\alpha,\beta}(v) = -\xi f^{\pi}_{\alpha,\beta}(v) + \xi^2 f^{2\pi/3}_{-\beta_1,-\beta_{-1}}(v).$$

Finally, suppose that $j = -1$. In this case $w_{\beta_1-\beta_i} = v_{\beta_1-\beta_i}$ for every $i \neq \pm 1$, and $w_{\beta_1} = v_{\beta_1}$. We obtain

$$\begin{aligned} f^{2\pi/3}_{\alpha,\beta}(w) - f^{2\pi/3}_{\alpha,\beta}(v) &= \sum_{i \neq \pm 1} N_{\beta_1,-\beta_i} v_{\beta_1-\beta_i} \xi N_{\beta_{-1},\beta_i-\beta_{-1}} v_{\beta_i-\beta_{-1}} \\ &\quad - v_{\beta_1} \xi \langle \beta_{-1}, \beta_{-1} \rangle v_{-\beta_{-1}} \\ &= -2\xi f^{\pi/2}_{\beta_1,-\beta_{-1}}(v). \end{aligned}$$

- **The π-equation.** First suppose that $j \neq \pm 1$. Then $w_{\beta_1-\beta_i} = v_{\beta_1-\beta_i}$ for $i \neq -j$, and $w_{\beta_i-\beta_1} = v_{\beta_i-\beta_1}$ for $i \neq j$. Moreover, $w_{-\beta_i} = v_{-\beta_i}$ for $i \neq -j$. Therefore

$$\begin{aligned} f^{\pi}_{\alpha,\beta}(w) - f^{\pi}_{\alpha,\beta}(v) = &- \sum_{\substack{i \neq \pm 1 \\ i \neq \pm j}} v_{-\beta_i} \xi N_{\beta_j,\beta_i-\beta_j} v_{\beta_i-\beta_j} \\ &+ v_{\beta_1-\beta_j} \xi N_{\beta_j,-\beta_1} v_{-\beta_1} \\ &+ v_{-\beta_j} \left(\xi^2 v_{-\beta_j} + \sum_{s=1}^{l} \langle \beta_j, \alpha_s \rangle \xi \widehat{v}_s \right) \\ &+ \xi N_{\beta_j,-\beta_{-1}} v_{-\beta_{-1}} v_{\beta_{-j}-\beta_1} \\ &- \xi N_{\beta_j,-\beta_j-\beta_{-j}} v_{-\beta_j-\beta_{-j}} v_{\beta_{-j}} \\ &- \sum_{s=1}^{l} \langle \beta_1, \alpha_s \rangle \widehat{v}_s \xi v_{-\beta_j} \\ &- \sum_{s=1}^{l} \langle \beta_{-1}, \alpha_s \rangle \widehat{v}_s \xi v_{-\beta_j} \\ &- \xi^2 v_{-\beta_j} v_{-\beta_j} . \end{aligned}$$

The terms containing ξ^2 cancel each other out. Arguing as above, it is not hard to see that

$$f^{\pi}_{\alpha,\beta}(w) - f^{\pi}_{\alpha,\beta}(v) = \xi f^{2\pi/3}_{-\beta_j,\beta_{-j}}(v).$$

For $j = 1$ we have $w_{\beta_i-\beta_1} = v_{\beta_i-\beta_1}$ and $w_{-\beta_i} = v_{-\beta_i}$ for all $i \neq \pm 1$. Moreover, by Lemma 9, we have

$$\sum_{s=1}^{l} \langle \beta_{-1}, \alpha_s \rangle \widehat{w}_s = \sum_{s=1}^{l} \langle \beta_{-1}, \alpha_s \rangle \widehat{v}_s ,$$

since $\langle\beta_{-1},\beta_1\rangle=0$.
Thus

$$\begin{aligned}f^{\pi}_{\alpha,\beta}(w)-f^{\pi}_{\alpha,\beta}(v)&=\sum_{i\neq\pm1}\xi N_{\beta_1,-\beta_i}v_{-\beta_i}v_{\beta_i-\beta_1}\\&-\sum_{i\neq\pm1}v_{-\beta_i}\xi N_{\beta_1,\beta_i-\beta_1}v_{\beta_i-\beta_1}\\&-\xi\langle\beta_1,\beta_1\rangle v_{-\beta_1}\cdot\sum_{s=1}^{l}\langle\beta_{-1},\alpha_s\rangle\widehat{v}_s\\&=-2\xi f^{2\pi/3}_{-\beta_1,-\beta_{-1}}(v).\end{aligned}$$

Finally, suppose that $j=-1$. Then $w_{\beta_1-\beta_i}=v_{\beta_1-\beta_i}$ and $w_{-\beta_i}=v_{-\beta_i}$ for all $i\neq\pm1$. Moreover, by Lemma 9, we have $\sum_{s=1}^{l}\langle\beta_1,\alpha_s\rangle\widehat{w}_s=\sum_{s=1}^{l}\langle\beta_1,\alpha_s\rangle\widehat{v}_s$, since $\langle\beta_1,\beta_{-1}\rangle=0$. Thus

$$\begin{aligned}f^{\pi}_{\alpha,\beta}(w)-f^{\pi}_{\alpha,\beta}(v)&=\sum_{i\neq\pm1}v_{\beta_1-\beta_i}\xi N_{\beta_{-1},-\beta_{-i}}v_{-\beta_{-i}}\\&-\sum_{i\neq\pm1}v_{-\beta_i}\xi N_{\beta_{-1},\beta_i-\beta_{-1}}v_{\beta_i-\beta_{-1}}\\&-\sum_{s=1}^{l}\langle\beta_1,\alpha_s\rangle\widehat{v}_s\cdot\xi\langle\beta_{-1},\beta_{-1}\rangle v_{-\beta_{-1}}\\&=-2\xi f^{2\pi/3}_{-\beta_{-1},-\beta_1}(v).\end{aligned}$$

(2) Suppose that $(\rho,\Omega)=\pi$. This means that $\rho=-\beta_j$ for some j.
- **The $\pi/2$-equation.** Note that $\beta_i-\rho=\beta_i+\beta_j$ is never a root, hence by Lemma 8 we have $w_{\beta_i}=v_{\beta_i}$ for all i, and $f^{\pi/2}_{\alpha,\beta}(w)=f^{\pi/2}_{\alpha,\beta}(v)$.
- **The $2\pi/3$-equation.** Here we have $w_{\beta_i}=v_{\beta_i}$, $w_{\beta_1}=v_{\beta_1}$.
If $j\neq\pm1$, then $w_{\beta_1-\beta_i}=v_{\beta_1-\beta_i}$ for $i\neq j$. Using Lemma 9, we get

$$f^{2\pi/3}_{\alpha,\beta}(w)=f^{2\pi/3}_{\alpha,\beta}(v)+N_{\beta_1,-\beta_j}N_{-\beta_j,\beta_1}\xi v_{\beta_1}v_{\beta_j}-v_{\beta_1}\xi\langle\beta_{-1},-\beta_j\rangle v_{\beta_j}.$$

It remains to note that $N_{\beta_1,-\beta_j}=-N_{-\beta_j,\beta_1}$ and $\langle\beta_{-1},-\beta_j\rangle=-1$, so that $f^{2\pi/3}_{\alpha,\beta}(w)=f^{2\pi/3}_{\alpha,\beta}(v)$.
For $j=1$ we have $w_{\beta_1-\beta_i}=v_{\beta_1-\beta_i}$ for all i, and (by Lemma 9)

$$\sum_{s=1}^{l}\langle\beta_{-1},\alpha_s\rangle\widehat{w}_s-\sum_{s=1}^{l}\langle\beta_{-1},\alpha_s\rangle\widehat{v}_s=\xi\langle\beta_{-1},-\beta_1\rangle v_{\beta_1}=0,$$

so that $f^{2\pi/3}_{\alpha,\beta}(w)=f^{2\pi/3}_{\alpha,\beta}(v)$ again.
Finally, if $j=-1$, then

$$w_{\beta_1-\beta_i}=v_{\beta_1-\beta_i}+N_{-\beta_{-1},\beta_{-i}}\xi v_{\beta_{-i}}$$

and

$$\sum_{s=1}^{l}\langle\beta_{-1},\alpha_s\rangle\widehat{w}_s-\sum_{s=1}^{l}\langle\beta_{-1},\alpha_s\rangle\widehat{v}_s=\xi\langle\beta_{-1},-\beta_{-1}\rangle v_{\beta_{-1}}=-2\xi v_{\beta_{-1}}.$$

Therefore

$$\begin{aligned}f^{2\pi/3}_{\alpha,\beta}(w) - f^{2\pi/3}_{\alpha,\beta}(v) &= \sum_{i\neq\pm1} N_{\beta_1,-\beta_i}N_{-\beta_{-1},\beta_{-i}}\xi v_{\beta_{-i}}v_{\beta_i} + 2\xi v_{\beta_1}v_{\beta_{-1}}\\ &= -\sum_{i\geq 2} 2\xi N_{\beta_1,-\beta_i}N_{\beta_{-1},-\beta_{-i}} + 2\xi v_{\beta_1}v_{\beta_{-1}}\\ &= 2\xi f^{\pi/2}_{\beta_1,\beta_{-1}}(v).\end{aligned}$$

- **The π-equation.** First suppose that $j \neq \pm 1$. Then $w_{\beta_1-\beta_i} = v_{\beta_1-\beta_i}$ for all $i \neq j$, and $w_{\beta_i-\beta_1} = v_{\beta_i-\beta_1}$ for all $i \neq -j$. Moreover, $w_{\beta_i} = v_{\beta_i}$ for all i. This means that

$$\begin{aligned}f^{\pi}_{\alpha,\beta}(w) - f^{\pi}_{\alpha,\beta}(v) = \sum_{\substack{i\neq\pm1\\ i\neq\pm j}} &(-\xi N_{-\beta_j,\beta_j-\beta_i}v_{\beta_j-\beta_i}v_{\beta_i})\\ &+ \xi N_{-\beta_j,\beta_1}v_{\beta_1}v_{\beta_j-\beta_1}\\ &+ v_{\beta_1-\beta_{-j}}\xi N_{-\beta_j,\beta_{-1}}v_{\beta_{-1}}\\ &- \left(-\sum_{s=1}^{l}\langle-\beta_j,\alpha_s\rangle\xi\widehat{v}_s - \xi^2 v_{\beta_j}\right)v_{\beta_j}\\ &- \sum_{s=1}^{l}\langle\beta_1,\alpha_s\rangle\widehat{v}_s\xi\langle\beta_{-1},-\beta_j\rangle v_{\beta_j}\\ &- \sum_{s=1}^{l}\langle\beta_{-1},\alpha_s\rangle\widehat{v}_s\xi\langle\beta_1,-\beta_j\rangle v_{\beta_j}\\ &- \xi^2 v_{\beta_j}v_{\beta_j}.\end{aligned}$$

The last four lines sum up to

$$-\sum_{s=1}^{l}\langle-\beta_1 - b_{-1} + \beta_j,\alpha_s\rangle\xi\widehat{v}_s v_{\beta_j} = -\xi v_{\beta_j}\sum_{s=1}^{l}\langle-\beta_{-j},\alpha_s\rangle\widehat{v}_s.$$

Applying Lemma 6 and noticing that $N_{-\beta_j,\beta_j-\beta_i} = -N_{\beta_j,\beta_i-\beta_j}$ we finally obtain

$$f^{\pi}_{\alpha,\beta}(w) - f^{\pi}_{\alpha,\beta}(v) = \xi f^{2\pi/3}_{\beta_j,-\beta_{-j}}(v).$$

Next, suppose that $j = 1$. Arguing exactly like in case (1), we get

$$f^{\pi}_{\alpha,\beta}(w) - f^{\pi}_{\alpha,\beta}(v) = 2\xi f^{2\pi/3}_{\beta_1,-\beta_{-1}}(v).$$

Similarly, for $j = -1$,

$$f^{\pi}_{\alpha,\beta}(w) - f^{\pi}_{\alpha,\beta}(v) = -2\xi f^{2\pi/3}_{\beta_{-1},\beta_1}(v).$$

(3) Suppose that $\angle(\rho,\Omega) = \pi/2$. This means that for some j we have $(\rho,\beta_j) = 1/2$ and $(\rho,\beta_{-j}) = -1/2$ (or vice versa). Note that $\beta_j - \rho$ and $\beta_{-j} + \rho$ are orthogonal roots with sum $\beta_{-j} + \beta j$; therefore they lie in Ω. By Chevalley commutator's formula we have $x_\rho(\xi) = [x_{\beta_{-j}+\rho}(\xi), x_{-\beta_{-j}}(\pm1)]$. Thus we reduce the question to two previous cases, since $\angle(\beta_{-j}+\rho,\Omega) = 0$ and $\angle(-\beta_{-j},\Omega) = \pi$.

(4) Suppose that $(\rho,\Omega) = 2\pi/3$. This means that for every i one of the scalar products (ρ,β_i), (ρ,β_{-i}) equals 0, while the other equals $-1/2$.
   - **The $\pi/2$-equation.** Note that $(\beta_i,\rho) \leq 0$ for every i. Hence $w_{\beta_i} = v_{\beta_i}$ for every i and we obtain $f^{\pi/2}_{\alpha,\beta}(w) = f^{\pi/2}_{\alpha,\beta}(v)$.

- **The $2\pi/3$-equation.** As above, we have $w_{\beta_i} = v_{\beta_i}$ for every i.
  If $(\beta_1, \rho) = -1/2$, then $w_{\beta_1-\beta_i} = v_{\beta_1-\beta_i}$ for every $i \neq \pm 1$, and $\langle \beta_{-1}, \rho \rangle = 0$, so that $\sum_{s=1}^{l} \langle \beta_{-1}, \alpha_s \rangle \widehat{w}_s = \sum_{s=1}^{l} \langle \beta_{-1}, \alpha_s \rangle \widehat{v}_s$. It follows that $f^{2\pi/3}_{\alpha,\beta}(w) = f^{2\pi/3}_{\alpha,\beta}(v)$.
  Now we assume that $(\beta_1, \rho) = 0$ and $(\beta_{-1}, \rho) = -1/2$. Then $w_{\beta_1-\beta_i} = v_{\beta_1-\beta_i}$ whenever $(\beta_i, \rho) = 0$. We obtain

$$f^{2\pi/3}_{\alpha,\beta}(w) - f^{2\pi/3}_{\alpha,\beta}(v) = \sum_{i\colon (\beta_i,\rho)=-1/2} N_{\beta_1,-\beta_i} \xi N_{\rho,\beta_1-\beta_i-\rho} v_{\beta_1-\beta_i-\rho} v_{\beta_i} + \xi v_{\beta_1} v_{-\rho}$$

  Note that exactly half of $2k-2$ indices $i = 2, \ldots, -2$ satisfy the condition $(\beta_i, \rho) = -1/2$, and for each one of them we have $(\beta_1 - \beta_i - \rho) + \beta_i = \beta_1 - \rho$. Hence the roots $\beta_1, -\rho, \{(\beta_1 - \beta_i - \rho), \beta_i\}_{i\colon (\beta_i,\rho)=-1/2}$ form a maximal square. Therefore

$$f^{2\pi/3}_{\alpha,\beta}(w) - f^{2\pi/3}_{\alpha,\beta}(v) = \xi f^{\pi/2}_{\beta_1,-\rho}(v).$$

- **The π-equation.** As above, note that for exactly half of the indices $i = 1, \ldots, -1$ we have $(\beta_i, \rho) = -1/2$, and for the other half we have $(\beta_i, \rho) = 0$. Put $J = \{i \mid (\beta_i, \rho) = -1/2\}$, $K = \{i \mid (\beta_i, \rho) = 0\}$. We know that $i \in J$ if and only if $-i \in K$. Again, we have $w_{\beta_i} = v_{\beta_i}$ for all i, and $w_{-\beta_i} = v_{-\beta_i}$ for $i \in K$.
  First suppose that $1 \in J$. Then $w_{\beta_1-\beta_i} = v_{\beta_1-\beta_i}$ for all i, and $w_{\beta_i-\beta_1} = v_{\beta_i-\beta_1}$ for $i \in J$. In this case we have

$$\begin{aligned} f^{\pi}_{\alpha,\beta}(w) - f^{\pi}_{\alpha,\beta}(v) = &\sum_{i\in K\setminus\{-1\}} v_{\beta_1-\beta_i} \xi N_{\rho,\beta_i-\beta_1-\rho} v_{\beta_i-\beta_1-\rho} \\ &- \sum_{i\in J\setminus\{1\}} \xi N_{\rho,-\beta_i-\rho} v_{-\beta_i-\rho} v_{\beta_i} \\ &+ \xi v_{-\rho} \cdot \sum_{s=1}^{l} \langle \beta_{-1}, \alpha_s \rangle \widehat{v}_s \end{aligned}$$

  It is easy to see that the roots $\beta_{-1}, -\rho, \{\beta_i - \beta_1 - \rho, \beta_{-i}\}_{i\in K\setminus\{-1\}}$ form a maximal square. It follow that

$$f^{\pi}_{\alpha,\beta}(w) - f^{\pi}_{\alpha,\beta}(v) = -\xi f^{2\pi/3}_{-\rho,\beta_{-1}}(v).$$

  Finally, suppose that $1 \in K$. Then $w_{\beta_1-\beta_i} = v_{\beta_1-\beta_i}$ for $i \in K$, and $w_{\beta_i-\beta_1} = v_{\beta_i-\beta_1}$ for all i. Similarly,

$$\begin{aligned} f^{\pi}_{\alpha,\beta}(w) - f^{\pi}_{\alpha,\beta}(v) = &\sum_{i\in J\setminus\{-1\}} \xi N_{\rho,\beta_1-\beta_i-\rho} v_{\beta_1-\beta_i-\rho} v_{\beta_i-\beta_1} \\ &- \sum_{i\in J\setminus\{-1\}} \xi N_{\rho,-\beta_i-\rho} v_{-\beta_i-\rho} v_{\beta_i} \\ &+ \sum_{s=1}^{l} \langle \beta_1, \alpha_s \rangle \widehat{v}_s \cdot \xi v_{-\rho} \end{aligned}$$

  It is easy to see that the roots $\beta_1, -\rho, \{\beta_1 - \beta_i - \rho, \beta_i\}_{i\in J\setminus\{-1\}}$ form a maximal square. It follows that

$$f^{\pi}_{\alpha,\beta}(w) - f^{\pi}_{\alpha,\beta}(v) = -\xi f^{2\pi/3}_{-\rho,\beta_1}(v).$$

(5) Suppose that $(\rho, \Omega) = \pi/3$. This means that $(\rho, \beta_1) = 0$ and $(\rho, \beta_{-1}) = \pi/3$ (or vice versa). Then $\rho - \beta_{-1}$ is a root and by Chevalley commutator's formula we have $x_\rho(\xi) = [x_{\rho-\beta_{-1}}(\xi), x_{\beta_{-1}}(\pm 1)]$. Thus we reduce the problem to previously discussed cases, since $\angle(\rho - \beta_{-1}, \Omega) = 2\pi/3$ and $\angle(\beta_{-1}, \Omega) = 0$.

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